A decentralized fault detection and isolation strategy for networked robots

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Abstract—This paper presents a distributed Fault Detection and Isolation (FDI) strategy, applied in conjunction with a distributed controller-observer schema, for a team of networked robots. Differently from other works in literature, the proposed FDI approach makes each robot of the team able to detect and isolate input faults of other robots even if not directly connected to it. The residual dynamics of the FDI observers are analytically investigated, and adaptive thresholds are derived to avoid the occurrence of false alarms in the presence of nonzero initial observer estimation errors. The approach is validated via numerical simulations in the case of time-varying centroid and formation control tasks.

I. INTRODUCTION

Teams of networked robots, i.e. multiple autonomous robots connected one to the others via a communication network, have been object of widespread research in the last years [11]. Such systems allow to take the advantages of distributed intelligence to achieve complex tasks, while each robot only uses local information and communicates with a sub-group of the robots in team called neighbors.

Recent researches in this field mainly focus on the development of distributed and cooperative control techniques where each robot interacts only with its neighbors and, together, they generate the global behavior of the team [6]. Such control algorithms often make use of graph theoretic methods that allow extracting analytical properties of the system on the base of the properties of the communication network they form [12]. Such analytical instruments are also used to deal with specific problems of networked systems as, e.g., consensus and connectivity maintenance. The consensus problem consists in how a set of robots can reach an agreement regarding a specific variable, exogenous or depending on the state of the single robots. Such problem has been recently investigated by a wide number of researchers; recent results are summarized in the book [17], while in references [14] and [18] the consensus algorithms are investigated with emphasis on robustness, time-delays and performance bounds. The control laws implemented on board of each individual robots often depend on the states of the other robots of the team; thus, the fault of one of them may jeopardize the proper execution of the task, and strategies for Fault Detection and Identification (FDI) become essential. The development of FDI scheme for networked robots is challenging because of the distributed nature of the system and of the limited information shared among the nodes. In the last years, many researchers have focused their attention on the fault diagnosis of distributed systems, proposing different centralized or decentralized approaches. In [19], a FDI scheme for networked systems is presented. In this solution, each subsystems periodically transmits to a centralized fault detection station the information about actuators and sensors. In [13], the diagnosis problem is formulated in terms of an isolability and both a centralized and decentralized architectures are developed and compared. In [16], an observer based schema is proposed for faults affecting both the local dynamics and the subsystems interconnections; in this case, each subsystem exchanges state information with all the other subsystems under certain conditions. In [10], Unknown Input Observers (UIOs) are proposed for FDI of networks of interconnected systems controlled with a decentralized control law. In the above cited papers, a necessary condition for the distributed detection of the fault of an agent by a healthy agent is that the two agents are able to communicate or sense each other.

In this paper, following the approach for single unit system developed in [7], we want to develop a distributed FDI algorithm for networked robots that allows each robot of the team to detect faults of other robots of team, even if not directly connected to it. The proposed strategy works in conjunction with a controller-observer schema derived from the one presented in [2], [5] and extended in [4]. Specifically, in the cited works each robot of the team was able to estimate the collective state of the system via a local observer that makes use only of information from the robot itself and from its direct neighbors. Here, starting from a modified version of the observer in [5], we make each robot able to detect additive input faults affecting any of the robots of the team. In the proposed solution, the fault detection and isolation functionalities require the definition of proper residuals. However, these residuals can be calculated without any information exchange between vehicles other than the one required by the observer-controller schema.

II. DECENTRALIZED OBSERVER-CONTROLLER SCHEME

Consider a system composed of \( N \) robots, where the \( i \)th robot’s state is denoted by \( x_i \in \mathbb{R}^n \). Each robot is characterized by a single-integrator dynamics

\[
\dot{x}_i = u_i + \phi_i, \tag{1}
\]
where \( u_i \in \mathbb{R}^n \) is the input vector and \( \phi_i \in \mathbb{R}^n \) is an additive fault term that is zero in healthy conditions. The collective state is given by \( x = [x_1^T \ldots x_N^T]^T \in \mathbb{R}^{Nn} \) and the collective dynamics is then expressed as

\[
\dot{x} = u + \phi,
\]

where \( u = [u_1^T \ldots u_N^T]^T \in \mathbb{R}^{Nn} \) is the collective input vector and \( \phi = [\phi_1^T \ldots \phi_N^T]^T \in \mathbb{R}^{Nn} \) is the collective fault vector.

The information exchange between the robots is described by a graph \( \Gamma(\mathcal{E}, \mathcal{V}) \) characterized by its topology [8],[9],[15], i.e., the set \( \mathcal{V} \) of the indexes labeling the \( N \) vertices (nodes), the set of edges (arcs) \( \mathcal{E} = \mathcal{V} \times \mathcal{V} \) connecting the nodes, and the \((N \times N)\) Laplacian matrix defined as

\[
L = \{ l_{ij} \} : \quad l_{ij} = \begin{cases} -1 & \text{if } (j,i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}, \quad l_{ii} = \sum_{j=1,j \neq i}^{N} -l_{ij},
\]

whose element \( l_{ij} \) (with \( j \neq i \)) is different form zero if the node \( j \)th can send information to node \( i \)th.

Moreover, we assume that the \( i \)th robot receives information from a reduced set of nodes (called its neighbors) \( \mathcal{N}_i = \{ j \in \mathcal{V} : (j,i) \in \mathcal{E} \} \), and it does not know the topology of the overall communication graph. Some properties and definitions about the communication graphs and used in the following are listed in [5] and, for the sake of brevity, omitted here. More details can be found in [12].

Following the approach presented in [2], [4], [5], the control architecture on board of each robot is composed of:

- a state observer providing an estimate, \( \hat{x} \in \mathbb{R}^{Nn} \), asymptotically convergent to the collective system state, \( x \), as \( t \to \infty \);
- a feedback control law, \( u_i = u_i(t, \hat{x}) \), such that one or more task function \( \sigma_i(t) \) asymptotically converge to a given reference, \( \sigma_i(t) \), as \( t \to \infty \);
- a fault detection and isolation schema able to detect a fault in the team and isolate the faulty vehicle.

A. The control objective and the feedback control law

The considered feedback control law is inherited from [4]; here, we recall its essential concepts.

The control objective is to make the team centroid and the relative formation follow desired time-varying references. To this aim, the two tasks are represented via the task functions:

- the centroid of the system:

\[
\sigma_1(x) = \frac{1}{N} \sum_{i=1}^{N} x_i = J_1x,
\]

where \( J_1 \in \mathbb{R}^{n \times Nn} \) is the Jacobian of the task.

- the formation of the system, expressed as an assigned set of relative displacement between the robots:

\[
\sigma_2(x) = [(x_2-x_1)^T \ldots (x_{N-N-1}^T)]^T = J_2x,
\]

where \( J_2 \in \mathbb{R}^{(N-1)n \times Nn} \) is the Jacobian of the task.

For the two task Jacobians, the following property holds:

\[
J_1^TJ_2^T = O_{n \times (N-1)n}, \quad J_2J_1^T = O_{(N-1)n \times n}, \quad (5)
\]

where \( J_i^T \) represent the pseudoinverses of the Jacobian matrices, and \( O_{p \times q} \) denotes the \((p \times q)\) null matrix. Eq. (5) expresses a compatibility condition (orthogonality) of the two tasks in the sense of [1]. The particular expressions of \( J_1 \) and \( J_2 \) can be found in [3] together with their pseudo-inverses.

In our decentralized solution, given the estimate \( \hat{x} \) of the system state \( x \), each robot compute an estimate of the collective input via the feedback control law:

\[
\dot{\hat{u}} = \dot{\hat{u}}_\sigma(t, \hat{x}) + \hat{u} \sigma_2(t, \hat{x})
\]

\[
= J_1^T \left[ \sigma_1(\cdot,d) + k_{1,c} (\sigma_1(\cdot,d) - \sigma_1(\hat{x}) \right] + J_2^T \left[ \sigma_2(\cdot,d) + k_{2,c} (\sigma_2(\cdot,d) - \sigma_2(\hat{x}) \right].
\]

The input vector \( u \) to robot \( i \) is computed selecting the relative component from \( \hat{u} \), i.e., \( u_i = \hat{u}_i \).

B. State observer

The state observer adopted in this paper is a modified version of the one presented in [5]; the modification is required to use the same observer both for control purposes and for the FDI strategy without increasing the information exchange burden.

Let \( \Gamma_i \) be the \((n \times Nn)\) matrix

\[
\Gamma_i = [O_n \ldots I_n \ldots O_n],
\]

and \( \Pi_i \) be the \((Nn \times Nn)\) matrix \( \Pi_i = I_n^T \Gamma_i \). It holds \( \sum_{i=1}^{Nn} \Pi_i = I_{Nn} \), and clearly, it is \( u_i = \Gamma_i^T \hat{u} \). The \( i \)th robot updates the estimate of the collective state via the observer

\[
\dot{\hat{x}} = k_o \left( \sum_{j=1}^{Nn} (\hat{y}_j - y_j) + \Pi_i (y_i - \hat{y}_i) \right) + \hat{u}_i,
\]

where \( k_o > 0 \) is a scalar gain, \( y = x - \int_{t_0}^{t} u(t)dt, \) and \( \hat{y}_j = \hat{x} - \int_{t_0}^{t} \hat{u}_j(t)dt \), where \( t_0 \) is the initial time instant. It worth noticing that \( \hat{y}_j \) depends on local information available to vehicle \( i \), and that each observer is updated using only the estimates \( \hat{y}_j \) received from direct neighbors. Thus, \( \hat{y}_j \in \mathbb{R}^{Nn} \) is the only information that is required to be exchanged among neighbors.

The collective estimation dynamics, in the absence of faults \( \phi_i = 0, i = 1, 2, \ldots, N \), is

\[
\dot{\hat{x}}^* = -k_o (L \otimes I_{Nn}) \hat{y}^* + k_o \Pi_t \hat{y}^* + \hat{u}^*,
\]

with \( \otimes \) denoting the Kronecker product operator and where

\[
\Pi_t = \text{diag} \left\{ \left[ \Pi_1 \ldots \Pi_N \right] \right\},
\]

\[
\hat{x}^* = \left[ \begin{array}{c} \hat{x}_1^* \ \hat{x}_2^* \ \ldots \ \hat{x}_{N}^* \end{array} \right]^T \in \mathbb{R}^{Nn} \text{ and } \hat{u}^* = \left[ \begin{array}{c} \hat{u}_1(t, \hat{x}) \ \hat{u}_2(t, \hat{x}) \ \ldots \ \hat{u}_N(t, \hat{x}) \end{array} \right] \in \mathbb{R}^{Nn}.
\]

In the following, we prove that, in absence of fault, \( \hat{x}^* \) exponentially converges to the origin with \( \hat{x}_i^* \) and \( \hat{u}_i \) the column vector with all elements set to 1.

To this aim, the exponentially convergence to the origin of \( \hat{y}^* \) is firstly proven with \( \hat{y}^* = \left[ \begin{array}{c} \hat{y}_1^* \ \ldots \ \hat{y}_{N}^* \end{array} \right] \in \mathbb{R}^{Nn} \) and \( \hat{y}_j = 1_N \otimes y - \hat{y}_j \in \mathbb{R}^{Nn} \).
**Theorem 1:** In presence of a directed strongly connected (or connected undirected) graph and in the absence of faults $(\phi_i = 0, i = 1, 2, \ldots, N)$, with the update law in eq. (8), $\hat{y}^*$ is exponentially convergent to the origin.

**Proof:** The proof simply comes from the fact that in presence of faults, the system in eq. (8) can be rearranged as
\[
\dot{\hat{y}}^* = -k_o(L \otimes I_N)\hat{y}^* + k_o\Pi^*\hat{y}^* = k_o\tilde{L}^*\hat{y}^*,
\]
with $\tilde{L}^* = L \otimes I_N + \Pi^*$. By noticing that from eq. (2) $\hat{y} = \hat{\phi}$, it holds
\[
\dot{\hat{y}}^* = -k_o\tilde{L}^*\hat{y}^* + \hat{\phi}^*\tag{10}
\]
with $\hat{\phi}^* = 1_N \otimes \phi$ and where the property $(L \otimes I_N)(1_N \otimes \phi) = (L1_N) \otimes (I_N \phi) = 0$ has been exploited. In [2], it was proved that matrix $-\tilde{L}^* = -L \otimes I_N - \Pi^*$ is Hurwitz provided that the graph is strongly connected. In this case, in absence of faults, $\phi^* = 0$, eq. (10) proves the theorem $\forall k_o > 0$. ■

**Theorem 2:** In the presence of a directed strongly connected (or connected undirected) graph and in the absence of faults, $\hat{\phi}_i (i = 1, 2, \ldots, N)$, with the update law in eq. (8), the stacked vector of the collective state estimation errors, $\tilde{\phi}^*$, is exponentially convergent to the origin.

**Proof:** See Appendix A

**Remark 2.1:** In [2] has been proven that the exponential stability of the observer leads to the exponential stability of the task errors $\hat{\sigma}_i$ $(l = 1, 2)$ with the control law in eq. (6). The proof relative to the convergence of $\hat{\phi}_i$ is omitted.

### III. Fault detection and isolation

In order to detect the occurrence of a fault, let us define for the robot $i$th $(i = 1, \ldots, N)$, the following residual vector
\[
\dot{r}_i = \sum_{j \in N_i} \left(\hat{\phi}_j \hat{y}_j - \hat{\phi}_i \hat{y}_i\right) + \Pi_i (y_i - \hat{y}_i)\tag{11}
\]
the above quantity does not require additional information exchange as makes use of the same quantities used in the local state observer. The stacked vector $r^* = [r^T_1 \ldots r^T_N]^T \in \mathbb{R}^{N \times N}$ can be expressed as
\[
r^* = -(L \otimes I_N)\hat{y}^* + \Pi^*\hat{y}^* = \tilde{L}^*\hat{y}^*,\tag{12}
\]
and, from eq. (10), its dynamics is given by
\[
\dot{r}^* = \tilde{L}^*\hat{y}^* = \tilde{L}^*\left(-k_o\tilde{L}^*\hat{y}^* + \hat{\phi}^*\right) = -k_o\tilde{L}^*r^* + \Pi^*\hat{\phi}^*\tag{13}
\]
From Theorem 1, it is straightforward derived that in the absence of faults, $r^*$ converges exponentially to zero, therefore the residuals of each robot $r_i$ converge to zero as well.

**A. Adaptive thresholds**

In presence of nonzero initial observer estimation errors, the residuals in eq. (11) can be different from zero as well, even in the absence of faults. To avoid the occurrence of false alarms, adaptive thresholds can be defined on the basis of the residual dynamics, then the decision about the occurrence of a fault is made when a residual exceeds its threshold. In order to choose the thresholds, let us derive from eq. (13) the dynamics of the residual $r_i$ in the absence of faults
\[
\dot{r}_i = -k_o\Pi_i r_i + k_o \sum_{j \in N_i} (r_j - r_i) = -k_o A' r_i + k_o \sum_{j \in N_i} r_j,\tag{14}
\]
where $A = \Pi_i + d_i I_N$, and $d_i$ is the dimension of $N_i$.

The residual $r_i = [r_1^T \ldots r_N^T]^T$ is such that each component $r_k \in \mathbb{R}^2$ represents the residual computed by the $i$th robot and referred to the $k$th robot.

From eq. (14), the expression of the residual $r_k$ is
\[
\dot{r}_k(t) = \chi_k(t-t_0) r_k(t_0) + \int_{t_0}^{t} k_o \chi_k(t-\tau) \sum_{j \in N_i} r_j d\tau,\tag{15}
\]
where $\chi_k(t) = \exp(-\eta_k I_t)$ and
\[
\eta_k = \left\{ \begin{array}{ll}
          k_o(1 + d_i) & \text{if } i = k \\
          k_o d_i & \text{if } i \neq k.
        \end{array} \right.
\]
The norm of $r_k$ can be upper bounded as follows
\[
\|r_k(t)\|^2 \leq \|r_k(t_0)\|^2 e^{-\eta_k (t-t_0)} + \frac{k_o d_i \zeta}{\eta_k - \lambda} \left( e^{-\lambda(t-t_0)} - e^{-\eta_k (t-t_0)} \right).\tag{16}
\]
By considering the exponential stability of $r^*$, it can be argued that there exist two constants, $\zeta$ and $\lambda$, such as $\|r_k(t)\| \leq \|r_k(0)\| e^{-\eta_k (t-t_0)} \leq \epsilon \exp(-\lambda(t-t_0))$, $\forall j, k = 1, \ldots, N$, therefore the following time-varying threshold, $\mu_k$ can be defined for the residual $r_k$
\[
\dot{\mu}_k(t) = \|r_k(t)\|^2 e^{-\eta_k (t-t_0)} + \frac{k_o d_i \zeta}{\eta_k - \lambda} \left( e^{-\lambda(t-t_0)} - e^{-\eta_k (t-t_0)} \right).\tag{17}
\]

The threshold calculation requires a reliable estimate of $\zeta$ and $\lambda$: the first one can be estimated on the basis of the information about the initial conditions of the system, while the latter can be estimated as the minimum eigenvalue of the matrix $L^*$ computed by considering the worst case for $L$.

**B. Residuals in the presence of faults**

If a fault on the $i$th robot occurs, i.e.,
\[
\dot{\phi} = \left[ 0^T \ldots \phi_i^T \ldots 0^T \right]^T,
\]
the following dynamics of $r_k, \forall k \in (1, \ldots, N)$, can be easily derived from eq. (13) and eq. (14)
\[
\dot{r}_k = -\eta_k I_t r_k + k_o \sum_{j \in N_i} r_k + \delta_k \phi_i,\tag{18}
\]
where $\delta_k$ is 1 if $i = k$, 0 otherwise. Since eq. (18) shows that the dynamics of all the residuals but $r_i$ are insensitive to the fault, a fault affecting the $i$th robot is detected and isolated by the $i$th robot itself if
\[
\exists t > t_0 : \|r_k(t)\| > \mu_k(t) \quad \forall k \in (1, \ldots, N), k \neq i, \forall t > t_0, \|r_i(t)\| \leq \mu_i(t).\tag{19}
\]
In the presence of the fault $\phi$, occurring at time $t_f > t_0$, the eq. (15) becomes

$$\dot{r}_k(t) = \chi_k(t-t_0)\cdot r_k(t_0) + \int_{t_0}^{t} k_i \chi_k(t-\tau) \sum_{j \in N_i} r_j d\tau$$

$$+ \int_{t_f}^{t} \chi_k(t-\tau) \delta_k \phi_i d\tau.$$  \hfill (20)

By straightforward calculations, omitted for the sake of brevity, the following sufficient detectability condition for the robot $i$th holds

$$\exists t > t_f : \left\| \int_{t_f}^{t} \chi_i(t-\tau) \phi_i d\tau \right\| \geq 2 \cdot \mu_i(t).$$  \hfill (21)

For the robots not affected by the fault, from eq. (13), due to the particular form of matrix $\Pi^*$, it can be seen that the residual dynamics does not directly depend on the fault vector, i.e., $\forall h, k \in (1, \ldots, N)$ and $h \neq i$

$$r_k = -h \iota I_n \cdot r_k + k_0 \sum_{j \in N_h} r_j.$$  \hfill (22)

If $k \neq i$, condition in eq. (19) implies that all the residuals $r_k$ remain below their thresholds. Moreover, since $\dot{r}_i$ depends on $\dot{r}_i$, directly (if $h \in N_i$) or indirectly through the residuals of the neighboring robots, the fault can be detected and isolated by the robot $h$ on the $i$th robot if

$$\exists t > t_f : \left\{ \begin{array}{l}
\forall k \in N, k \neq i, \forall t > t_0, \left\| r_k(t) \right\| > \mu_i(t), \\
\forall t > t_0, \left\| r_k(t) \right\| \leq \mu_i(t).
\end{array} \right.$$  \hfill (23)

A detectability sufficient condition for the robot $h$th not affected by the fault, is given by the following inequality

$$\exists t > t_f : \left\| \int_{t_0}^{t} k_i \chi_i(t-\tau) \sum_{j \in N_h} r_j d\tau \right\| \geq 2 \cdot \mu_i.$$  \hfill (24)

Condition in eq. (21) depends on $\phi_i$, on the contrary condition in eq. (24) depends only on the propagation of the fault signature via the residuals of the neighboring robots.

**Remark 3.1:** The residuals are decoupled in such a way that the fault $\phi_i$ affects only the residuals $r_i$. Such decoupled structure makes the proposed scheme effective also in the presence of multiple faults affecting different robots.

**IV. NUMERICAL SIMULATIONS**

In this section we present the results of a numerical simulation analysis performed to validate the effectiveness of the controller-observer and FDI strategies presented in the previous sections. In particular, we considered a team of five mobile robots, each of them implementing the control algorithm in eq. (6) and the observer in eq. (7), where the control and observer gains were set respectively to $k_c = 2$, $k_{1,c} = 1.3$, and $k_{2,c} = 1.3$. The network topology is assumed as a rigid directed graph reported in Figure 1.

The robots were commanded to keep a constantly linear formation with relative distance of 0.4 m, while the team centroid was commanded to follow a sinusoidal path.

During the motion, we induce the input failure to one of the robots applying a null input command ($\phi_2(t) = -u_2(t)$ for $t > 20$ s), i.e., the robot suddenly stops.

The information initially available to each robot is assumed to be limited to its position ($x_i(t_0)$) and to the number of robots in the team ($N$). Each local observer $\dot{x}_i$ is then initialized as all the robots were in the same position ($\dot{x}_i(t_0) = x_i(t_0)$). The robots are initially displaced inside a closed region of $2 \times 2$ m (centered in $[1.5, 2]$ m) following a uniformly random distribution. Assuming that the size of the area in which the robots are initially displaced is known, the previous initialization choices allow evaluating the maximum initial values of the residual errors, and using them in the residual adaptive thresholds computation.

Figure 2 shows the paths of the robots during the mission and the desired path of the team centroid. From the figure, it is possible noticing that robot 2 stops during the course of the mission due to the fault occurrence.

Figure 3 shows the plots of the norms of the estimation and task functions errors, and it is possible noticing that all the
If not directly connected to it; as an example, Figure 4 shows threshold values used for the fault detection. Figure 5 shows components of the different observers relative to robot 1.

Fig. 4. Solid line: Norm of the residual components with respect to robot 1 of the different robots. Dashed line: Threshold used for the fault detection. Occurrence. The same happens for the residuals of the other healthy vehicles are able to detect and isolate the faulty ones even in the case they are not direct neighbours. As future work, a fault accommodation schema that tries to recover the faulty vehicles or that exclude them from the team will be designed. Finally, experimental tests will be performed with a real team of networked robots.

V. CONCLUSIONS

In this paper, we present a distributed fault detection and isolation strategy, jointly working a distributed observer-controller scheme, for a team of networked robots. Based on the designed observer, a proper vector of residuals is introduced, and detectability and isolability conditions are analytically investigated. Differently from other works, healthy vehicles are able to detect and isolate the faulty ones even in the case they are not direct neighbours. As future work, a fault accommodation schema that tries to recover the faulty vehicles or that exclude them from the team will be designed. Finally, experimental tests will be performed with a real team of networked robots.

APPENDIX

A. Proof of Theorem 2

In absence of faults, the system in eq. (8) is

\[ \dot{\hat{x}}^* = k_o \tilde{L} \hat{y}^* + \hat{u}^*. \]  

(25)

As it is \( \dot{x}^* = 1_N \otimes \dot{x} = 1_N \otimes u = u^* \), the system in eq. (25) can be written as

\[ \dot{x}^* = -k_o \tilde{L} \hat{y}^* + \hat{u}^* = \hat{u}^* (t, \dot{x}^*) + d, \]  

(26)

with \( \hat{u}^* = 1_N \otimes u - \hat{u}^* = [1_u^T \ 2_u^T \ldots N_u^T]^T \in \mathbb{R}^{N^2 n} \). Provided that the conditions in Theorem 1 hold, the above system can be seen as a dynamical systems with \( d (d = -k_o \tilde{L} \hat{y}^*) \) as an exponentially vanishing input. As a consequence, the error \( \dot{x}^* \) in eq. (26) exponentially reaches the origin provided that the system

\[ \dot{x}^* = \hat{u}^* (t, \dot{x}^*) = \hat{u}^* \Pi_{i \neq l} (t, \dot{x}^*) + \hat{u}^* \Pi_{i \neq l} (t, \dot{x}^*) \]  

(27)

is exponentially stable. It is, thus, necessary to found an expression of \( \hat{u}^* \) as function of \( \hat{x}^* \) in the case of control law in eq. (6). At this end, it holds \((l = 1, 2)\):

\[ \hat{u}_{\sigma}^* = u_{\sigma} (t) - i \hat{u}_{\sigma} (t) = \sum_{k=1}^{N} \Pi_k \hat{u}_{\sigma} (t) - i \hat{u}_{\sigma} (t) \] 

\[ = k_{l,c} \left( J_l^1 J_l^1 \hat{x} - \sum_{k=1}^{N} \Pi_k J_l^1 J_l^k \hat{x} \right) \]  

(28)

thus by collecting the differences \( u_{\sigma} (t) - i \hat{u}_{\sigma} (t) \), \( \forall i \), it is

\[ \hat{u}_{\sigma_{l}}^* = 1_N \otimes u_{\sigma_{l}} - \hat{u}_{\sigma_{l}}^* = k_{l,c} \begin{bmatrix} J_l^1 J_l^1 \hat{x} - \sum_{k=1}^{N} \Pi_k J_l^1 J_l^k \hat{x} \\ \vdots \\ J_l^1 J_l^N \hat{x} - \sum_{k=1}^{N} \Pi_k J_l^1 J_l^k \hat{x} \end{bmatrix} \]  

(29)

Let us define the following matrices:

\[ J_l^i = \text{diag} \{ [J_l^1, J_l^1, J_l^1, \ldots, J_l^1, J_l^1] \} \in \mathbb{R}^{N^2 n \times N^2 n} \]  

(29)
\( \Pi^* = 1_N \otimes \begin{bmatrix} \Pi_1 & \Pi_2 & \ldots & \Pi_N \end{bmatrix} \in \mathbb{R}^{N^2 \times N^2}; \) \hspace{1cm} (30)\]

it is:

\[ 1_N \otimes u_{\sigma_i} - u^*_{\sigma_i} = k_{l,c}(I_{N^2 n} - \Pi^*)J_i^* \hat{x}^* \]. \hspace{1cm} (31)\]

In the case of control law in eq. (6), it is:

\[ 1_N \otimes u_{\sigma_i} - u^*_{\sigma_i} = (I_{N^2 n} - \Pi^*)(k_{1,c} J_1^* + k_{2,c} J_2^*) \hat{x}^* . \] \hspace{1cm} (32)\]

The properties listed below will be exploited in the following:

**Property 1:** \( \Pi^* (1_N \otimes v) = (1_N \otimes v) \);  
**Property 2:** as the matrix \((I_{N^2 n} - \Pi^*)\) is idempotent \( \text{rank}(I_{N^2 n} - \Pi^*) = N^2 n - N^2 n; \)  
**Property 3:** it comes from [3] \( J_1^*J_1 + J_2^*J_2 = I_{N^2 n}; \)  
**Property 4:** matrix \((I_{N^2 n} - \Pi^*)\) is symmetric and has \( N^2 n - N^2 n \) eigenvalues equal to 1 and \( N^2 n \) eigenvalues equal to 0. It is a direct consequence of Property 2 and of the fact that \((I_{N^2 n} - \Pi^*)\) is idempotent (idempotent matrices admits only 0 and 1 as eigenvalues).

Because of Property 1, it is straightforward to show that:

\[ (I_{N^2 n} - \Pi^*)(k_{1,c} J_1^* + k_{2,c} J_2^*) \hat{x}^* = -(I_{N^2 n} - \Pi^*)(k_{1,c} J_1^* + k_{2,c} J_2^*) \hat{x}^* \] \hspace{1cm} (33)\]

as

\[-(I_{N^2 n} - \Pi^*)(k_{1,c} J_1^* + k_{2,c} J_2^*)(1 \otimes x) = -(I_{N^2 n} - \Pi^*)(1 \otimes \left( (k_{1,c} J_1^*J_1 + k_{2,c} J_2^*J_2) x \right)) = 0_{N^2 n}. \]  

In sum, the system in eq. (27) becomes:

\[ \hat{x}^* = -(I_{N^2 n} - \Pi^*)(k_{1,c} J_1^* + k_{2,c} J_2^*) \hat{x}^* = A^* \hat{x}^*. \] \hspace{1cm} (34)\]

By assuming \( k_{1,c} = k_{2,c} = k_c > 0 \) and from Property 3,

\[ \hat{x}^* = -k_c(I_{N^2 n} - \Pi^*) \hat{x}^*. \] \hspace{1cm} (35)\]

From Property (4), it follows that the dynamic matrix of the above system is symmetric negative semidefinite. Thus, the solution to the above system is:

\[ \hat{x}^*(t) = e^{-k_c(I_{N^2 n} - \Pi^*)(t-t_0)} \hat{x}^*(t_0) \] \hspace{1cm} (36)\]

where \( \hat{x}^*(t_0) \) is the collection of the estimation errors at the initial time instant. As a final step, it is:

\[ \hat{x}^*_\infty = \lim_{t \to +\infty} \hat{x}^*(t) = \Pi^* \hat{x}^*(t_0), \] \hspace{1cm} (37)\]

where the property \( \lim_{t \to +\infty} e^{-At} = I - A \) for any idempotent matrix \( A \) was exploited. In sum:

\[ \hat{x}^*_\infty = \Pi^* \hat{x}^*(t_0) = (1_N \otimes [\Pi_1, \Pi_2, \ldots, \Pi_N]) \hat{x}^*(t_0) = 1_N \otimes \left( \begin{bmatrix} \Pi_1 & \Pi_2 & \ldots & \Pi_N \end{bmatrix} \hat{x}^*(t_0) \right) \] \hspace{1cm} (38)\]

Eq. (38) clearly states that \( \hat{x}^* = \lambda \hat{x}, \forall \lambda, j \) and, thus, all the estimate \( \hat{x}^* \) exponentially converges to a common value. Therefore, there is no evidence that this value is \( x \). In order to show that the consensus value is, indeed, \( x \), it is worth noticing that as a consequence of \( \tilde{y}_i = 0 \) at steady state,

\[ x = \int_{t_0}^{t} u \, dt - \int_{t_0}^{t} \tilde{u} \, d\tau, \forall i \] \hspace{1cm} (39)\]