Analysis and Synthesis of Manipulator Workspace

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1. Kinematic chain and H-D notation for manipulators

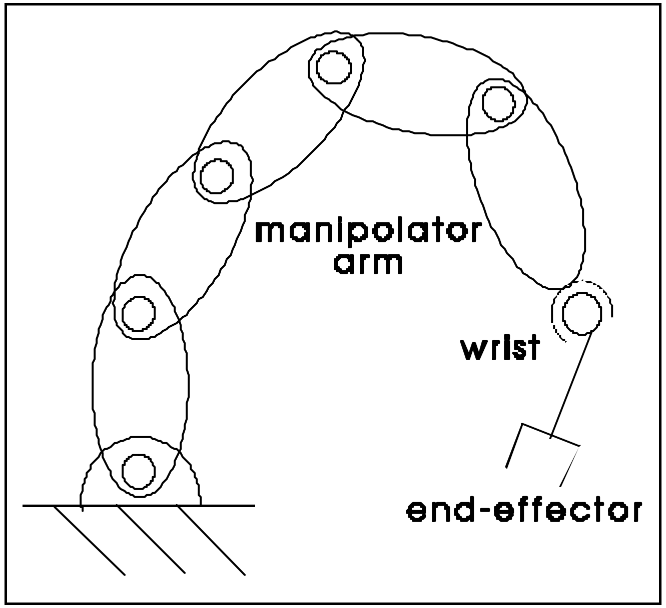
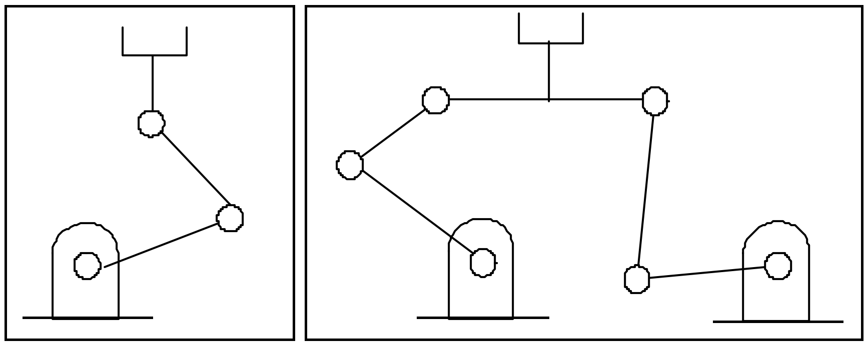


Fig.3.1: A scheme for the manipulator architecture of a robot with the arm, wrist, and end-effector.



a) b)

Fig.3.2: Planar examples of kinematic chains of manipulators: a) serial chain as open type; b) parallel chain as closed type.

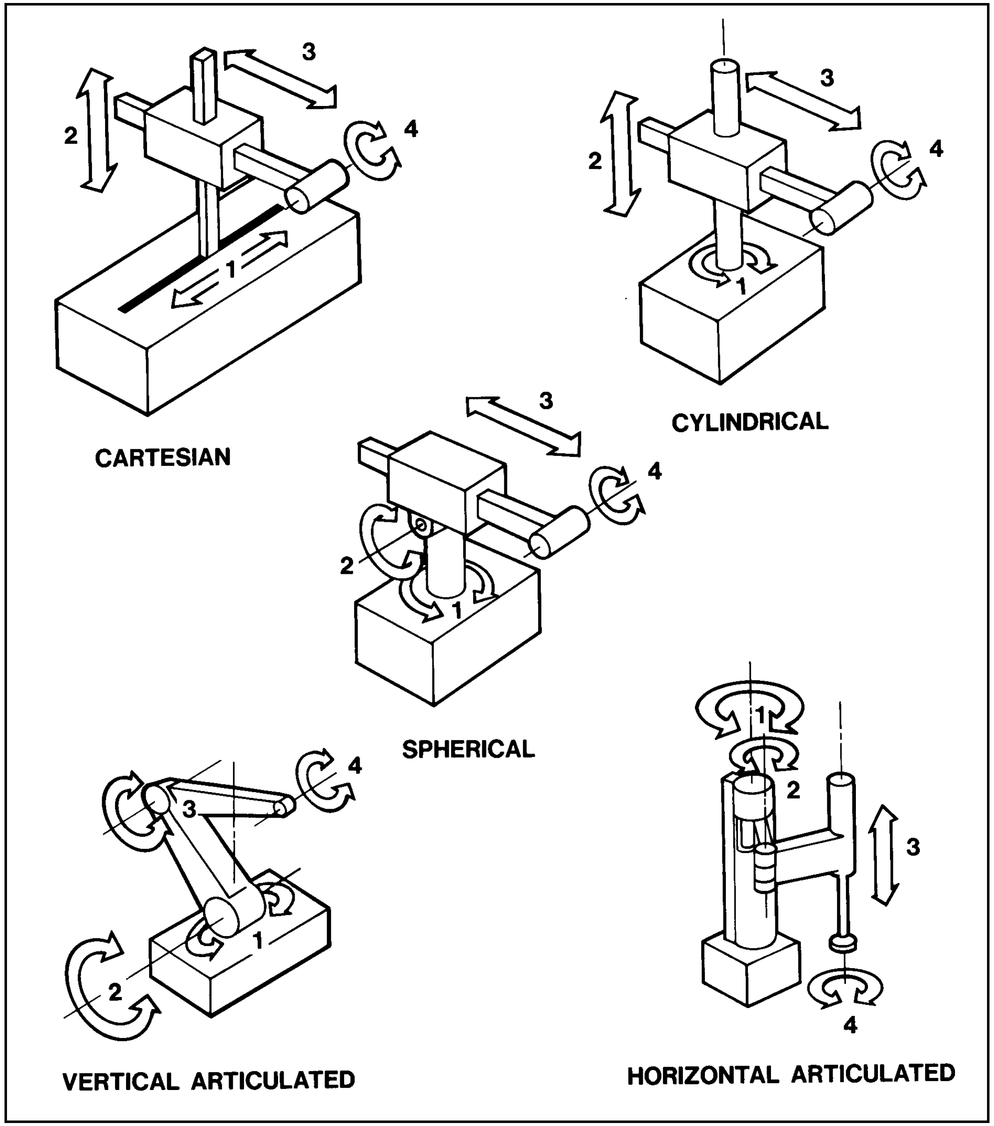
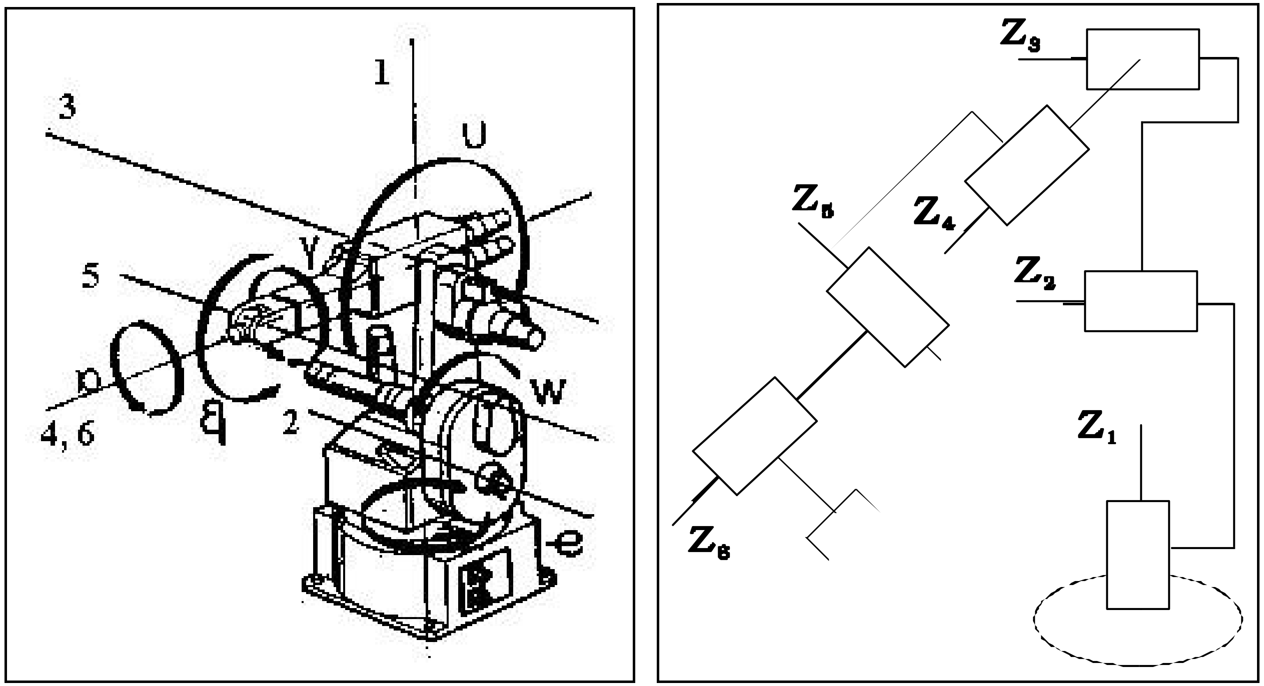


Fig.3.4: Manipulator architectures for industrial robots.

- A kinematic model of a manipulator can be named as functional when its scheme refers to kinematic parameters only, but permits also to understand the motion capability of the manipulator architecture.

A kinematic functional model can be determined from the mechanical design of a robot through the following step-by-step procedure:
- identification of the type of the joints;
- identification of the position of each joint axes;
- identification of the geometry of the links;
- drawing of a scheme for the kinematic chain.



a) b)

Fig.3.5: An example of modeling an industrial robot: a) the mechanical design; b) the corresponding kinematic functional scheme.

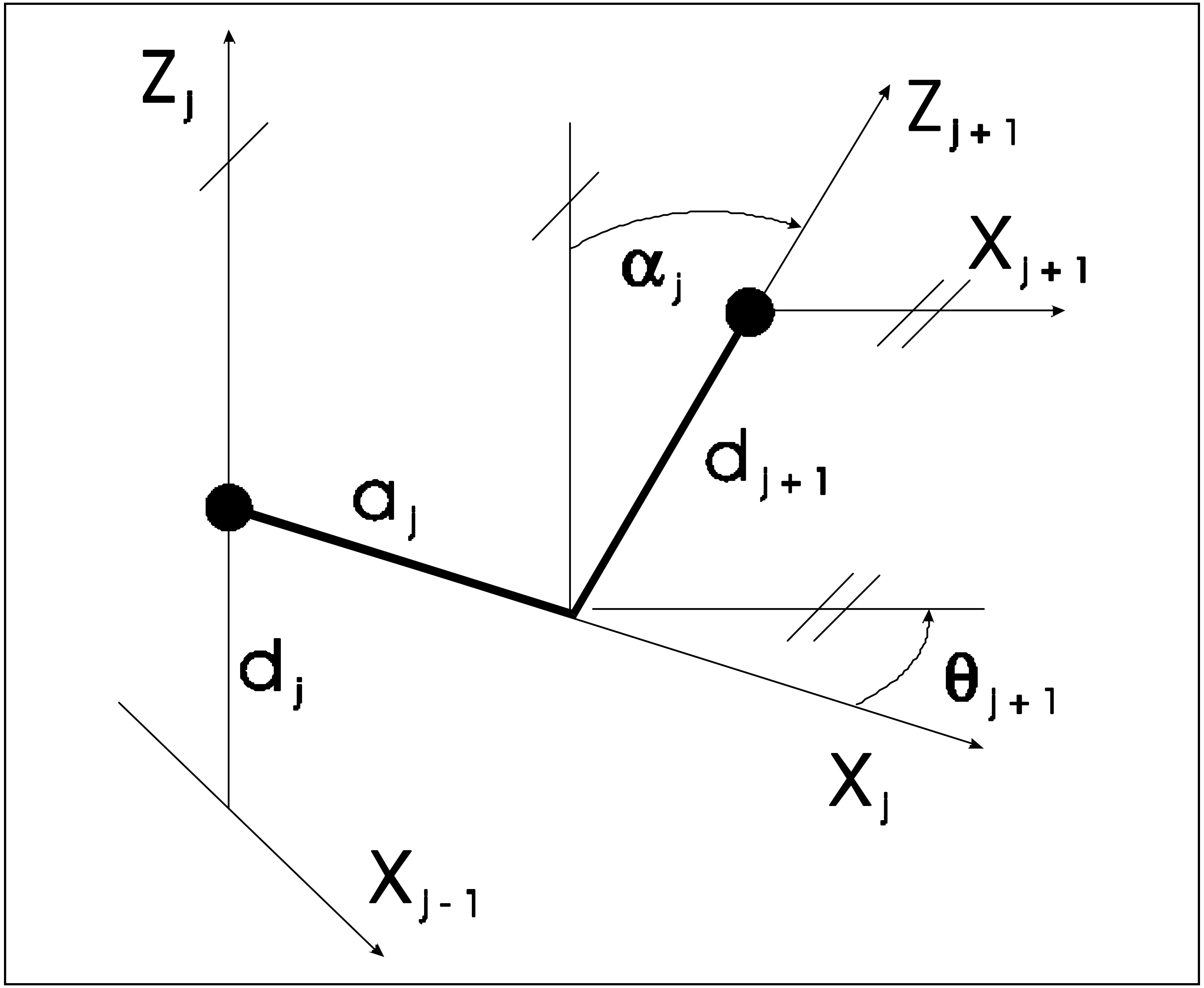
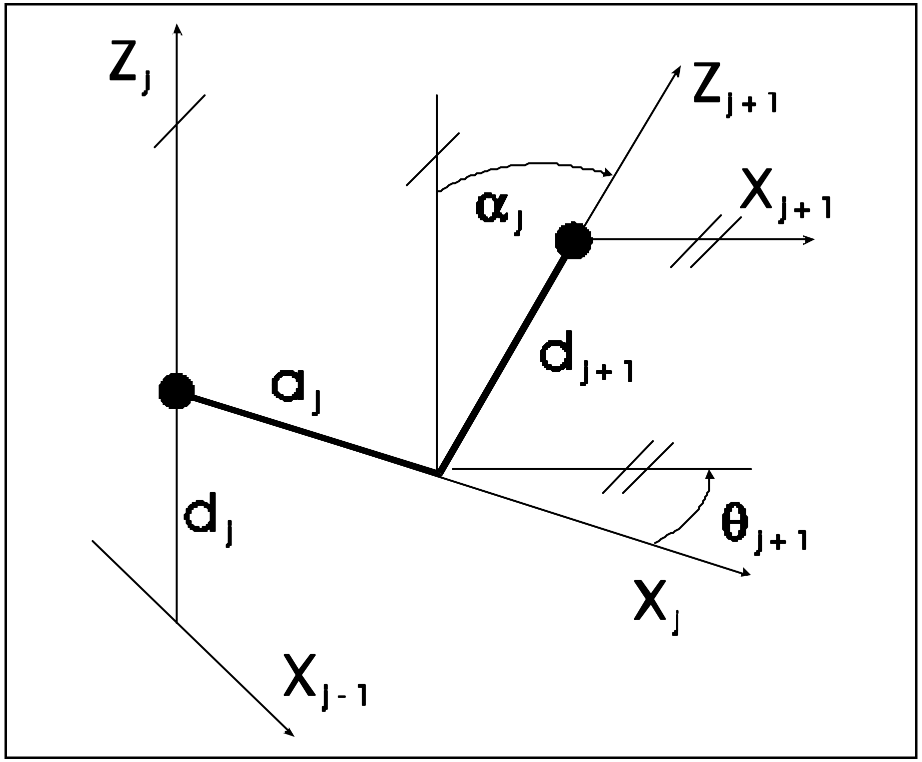


Fig.3.6: A kinematic scheme for manipulator link parameters according to the H-D notation.



The kinematic parameters of a manipulator can be defined according to the H-D notation in Fig.3.6 as:

- a_j, which is named as the link length that is measured as the distance between the Z_j and Z_{j+1} axes along X_j;
- α_j, which is named as the twist angle that is measured as the angle between the Z_j and Z_{j+1} axes about X_j;
- d_{j+1}, which is named as the link offset that is measured as the distance between X_j and X_{j+1} axes along Z_{j+1};
- θ_{j+1}, which is named as the joint angle that is measured as the angle between the X_j and X_{j+1} axes about Zj_{j+1}

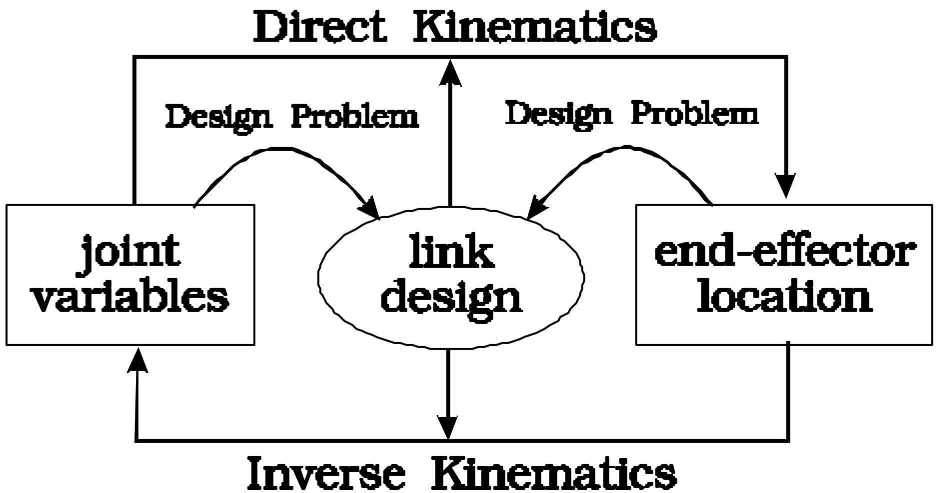
When a joint can be modeled as a rotation pair, the angle θ_{j+1} is the corresponding kinematic variable.
When a joint can be modeled as a prismatic pair, the distance d_{j+1} is the corresponding kinematic variable.

the position problem can be considered from different viewpoints depending of the unknowns:

- **Kinematic Direct Problem** in which the dimensions of a manipulator are given through the dimensional H-D parameters of the links but the position and orientation of the end-effector are determined as a function of the values of the joint variables;

- **Kinematic Inverse problem** in which the position and orientation of the end-effector of a given manipulator are given, and the configuration of the manipulator chain is determined by computing the values of the joint values.

- **Kinematic Indirect Problem** (properly **Design Problem**) in which a certain number of positions and orientations of the end-effector are given but the type of manipulator chain and its dimensions are the unknowns of the problem.



Transformation matrix

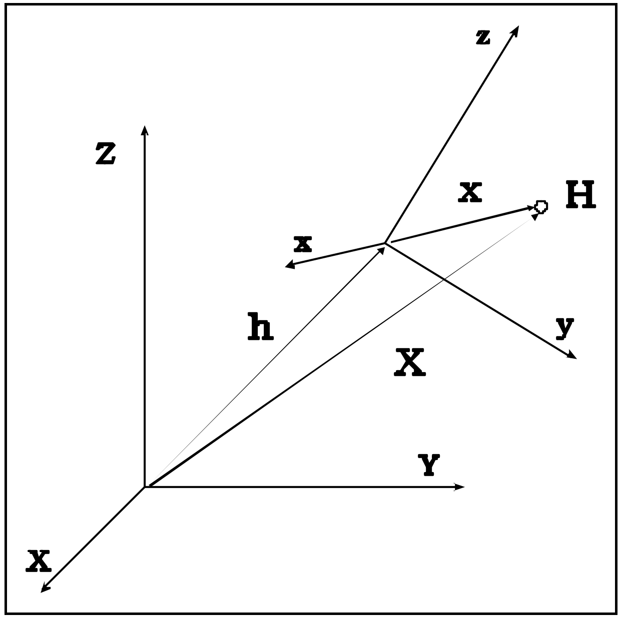


Fig.3.7: Vectors for relative position and orientation between two reference frames.

$$\mathbf{X} = \mathbf{h} + \mathrm{R}\mathbf{x} \qquad \mathbf{X} = \mathrm{T}\,\mathbf{x}$$

When the Transformation Matrix is defined as

$$\mathrm{T} = \begin{vmatrix} \mathrm{R} & \mathbf{h} \\ 0 & 1 \end{vmatrix}$$

Rotation Matrix:
$$\mathrm{R} = \begin{vmatrix} \mathrm{i}\cdot\mathrm{I} & \mathrm{i}\cdot\mathrm{J} & \mathrm{i}\cdot\mathrm{K} \\ \mathrm{j}\cdot\mathrm{I} & \mathrm{j}\cdot\mathrm{J} & \mathrm{j}\cdot\mathrm{K} \\ \mathrm{k}\cdot\mathrm{I} & \mathrm{k}\cdot\mathrm{J} & \mathrm{k}\cdot\mathrm{K} \end{vmatrix}$$

$$\mathbf{X}_0 = {}_0T_N \; \mathbf{X}_N$$

and therefore a resultant transformation matrix for a manipulator can be obtained as

$${}_0T_N = {}_0T_1 \; {}_1T_2 \; \ldots \; {}_{N\text{-}1}T_N = \prod_{k=0}^{N-1} {}_kT_{k+1}$$

This expression can be considered the fundamental typical formulation for the Direct Kinematic of manipulators.

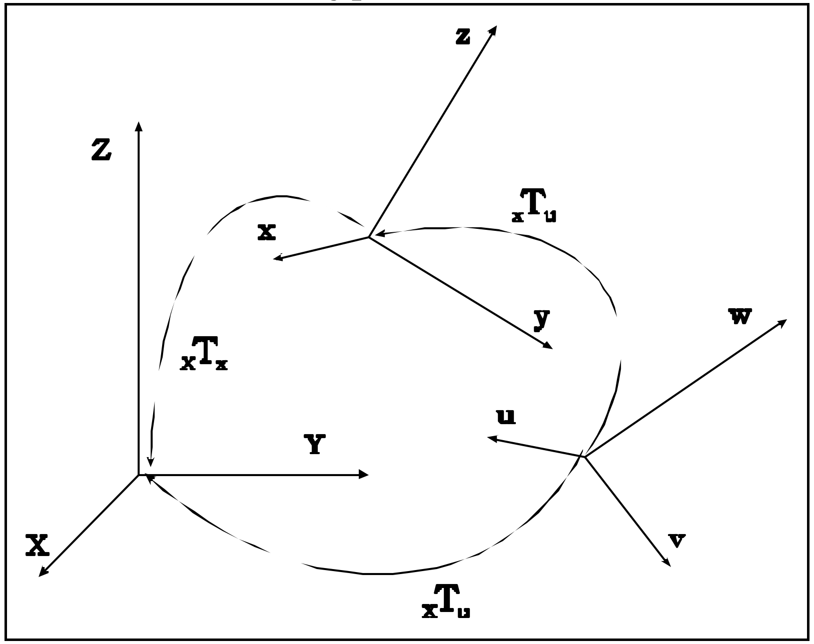


Fig.3.8: A scheme for indicating which frames a transformation matrix is formulated with respect to.

In particular, by using the above-mentioned properties and formulation for the transformation matrix, a general expression for ${}_kT_{k+1}$ can be given by referring to the general manipulator scheme of Fig.3.6, in the form

$$
{}_kT_{k+1} = \begin{vmatrix} \cos\theta_{k+1} & -\sin\theta_{k+1} & 0 & a_k \\ \sin\theta_{k+1}\cos\alpha_k & \cos\theta_{k+1}\cos\alpha_k & -\sin\alpha_k & -\sin\alpha_k\ d_{k+1} \\ \sin\theta_{k+1}\sin\alpha_k & \cos\theta_{k+1}\sin\alpha_k & \cos\alpha_k & \cos\alpha_k\ d_{k+1} \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad (3.1.15)
$$

as a composition of elementary matrices that describe elementary differences between reference frames that are described by the H-D parameters.

An elementary matrix corresponds to a rotation about a reference axis or a translation along such an axis.

Thus the matrix ${}_kT_{k+1}$ can be given by the expression

$$
{}_kT_{k+1} = \mathrm{Rot}(X_k, \alpha_k)\,\mathrm{Trasl}(X_k, a_k)\,\mathrm{Rot}(Z_{k+1}, \theta_{k+1})\,\mathrm{Trasl}(Z_{k+1}, d_{k+1}) \qquad (3.1.19)
$$

or alternatively by using elementary helicoidal motions, it is given by

$$
{}_kT_{k+1} = \mathrm{Screw}(X_k, a_k, \alpha_k)\,\mathrm{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1}) \qquad (3.1.20)
$$

Definition and Generation of workspace

- *The workspace W(H) is defined as the region of points that can be reached by a reference point H on the manipulator extremity. This is the Position Workspace.*

The workspace is defined as generated by a reference point H on the extremity of the manipulator chain that is moved to reach all possible positions because of mobility ranges of the joints.

- *Similarly, the Orientation Workspace can be defined as the set of orientations that can be reached by the manipulator extremity.*

Fundamental characteristics of the manipulator position workspace are recognized as:

- **the shape and volume** of the workspace, which is a solid of revolution for manipulators with only revolute joints, and can be a parallelepiped for manipulators with prismatic joints;
- **the hole**, whose existence is determined by a region of unreachable points that can individuate straight-lines surrounded by the workspace yet;
- **the voids,** which are regions of unreachable points that are buried within the workspace yet.

Similar characteristics can be identified for the orientation workspace that nevertheless has a different topology.

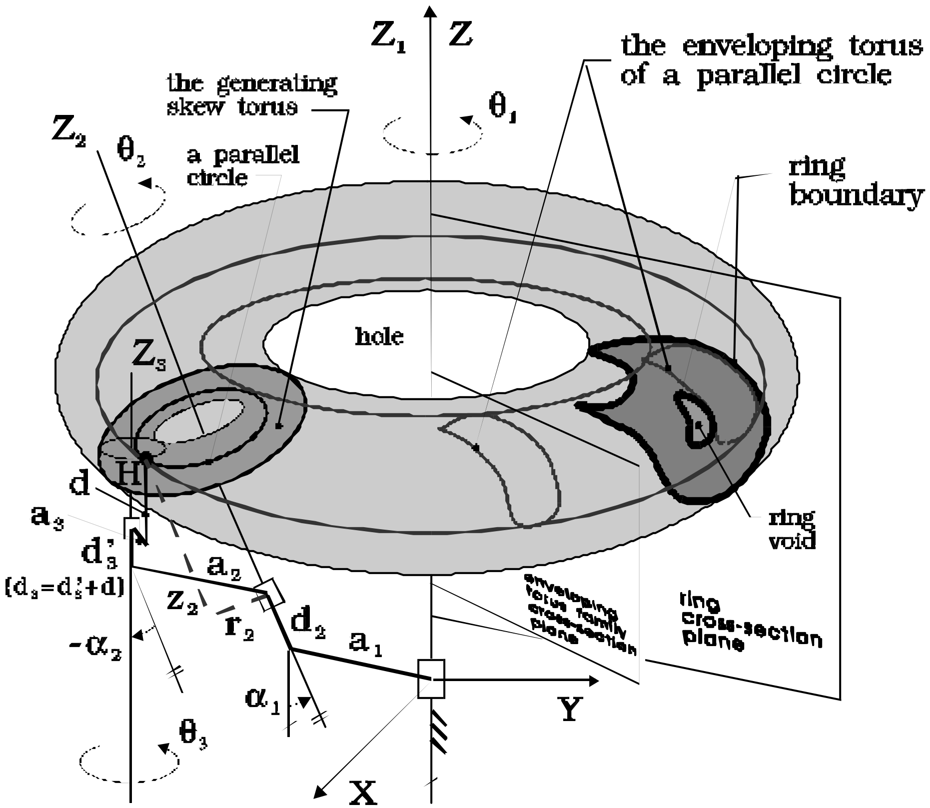


Fig.2 Design parameters and workspace geometry for 3R manipulators.

Revolving a torus about an axis generates a ring:

$$W_{3R}\left(H\right)=\bigcup_{\vartheta_1=0}^{2\pi} T_{R_2R_3}\left(H\right) \qquad (3.1.27)$$

Alternatively, a ring $W_{3R}(H)$ can be considered as the union of the tori $T_{R1R2}(H)$,

$$W_{3R}\left(H\right)=\bigcup_{\vartheta_3=0}^{2\pi} T_{R_1R_2}\left(H\right) \quad . \qquad (3.1.28)$$

Workspace generation

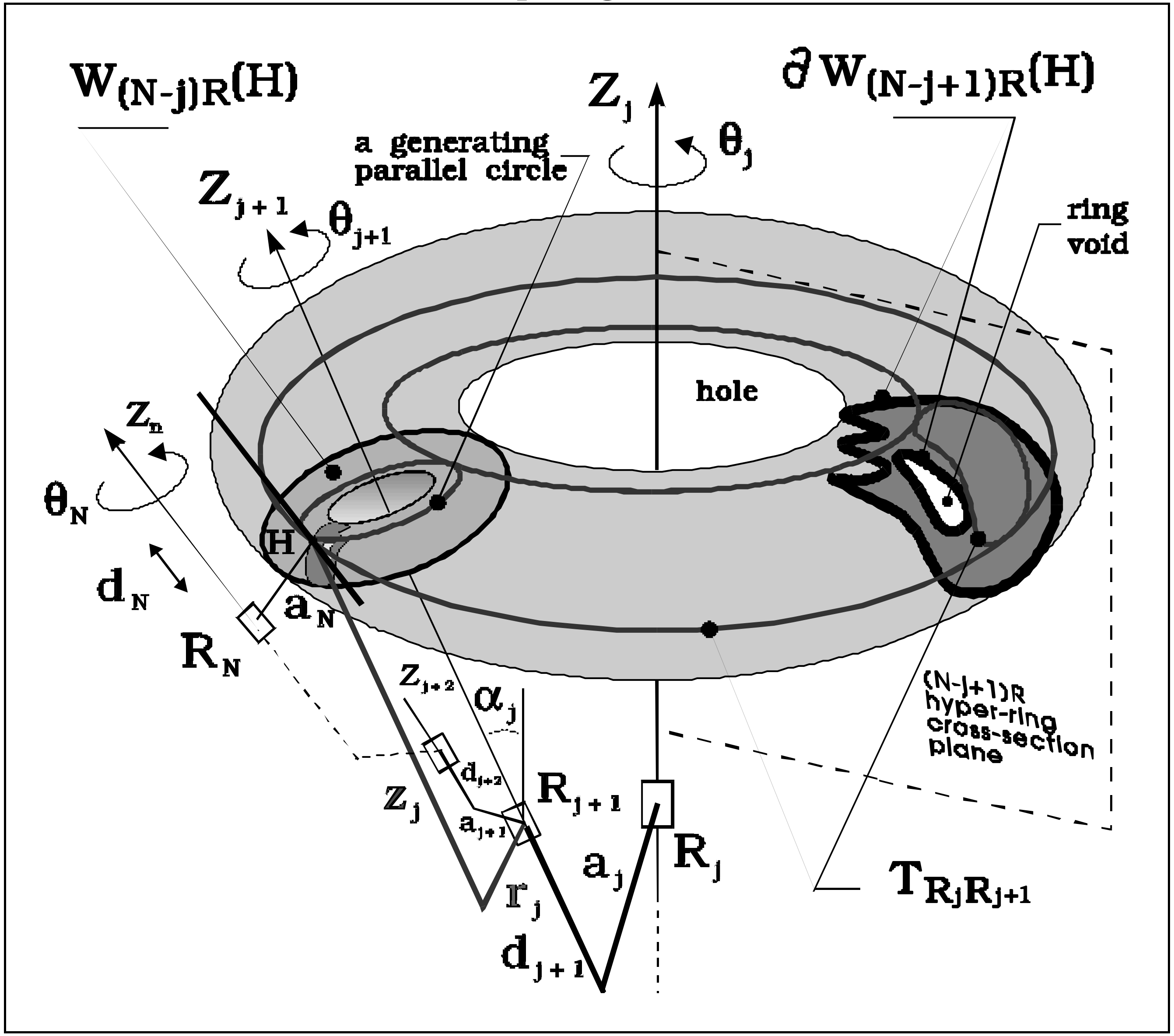


Fig.3.11: A descriptive view of workspace generation for serial manipulators with revolute joints.

The generation process of a hyper-ring

is a consecutive revolving process of a circle, a torus, a ring, a 4R hyper-ring, and so on.
This can be expressed through a revolution operator Rev in the form

$$W_{(N-j+1)R}(H) = \underset{\vartheta_j=0}{\overset{2\pi}{\mathrm{Rev}}}\, W_{(N-j)R}(H) \qquad j = 1,...,N-1 \qquad (3.1.30)$$

Alternatively,

$W_{(N-j+1)R}$ (H) can be considered as the union of a suitable torus family which is traced by the boundary points in the revolving torus, ring, 4R hyper-ring and so on, when they are rotated completely about the first two revolute axes in the corresponding generating sub-chain. It can be expressed in the form

$$W_{(N+1-j)R}(H) = \bigcup_{\vartheta_j,\vartheta_{j+1}=0}^{2\pi} T_{R_j R_{j+1}}\left[\partial W_{(N-j)R}(H)\right] \qquad (3.1.31)$$

where T_{RjRj+1} (H), represents a torus generated by revolutions θ_j and θ_{j+1} about the joints axes of R_j and R_{j+1}. The revolution in θ_{j+1} generates a parallel circle in $W_{(N-j)R}$ (H) and together with θ_j generates the torus T_{RjRj+1} (H).

the boundary $\partial W_{3R}(H)$ of a ring

can be thought as the envelope of torus surfaces generated by revolution of the generating torus or, alternatively, it can be obtained by an envelope of torus surfaces that are traced from the parallel circles of the generating torus. The latter procedure can be expressed according to Eq.(3.1.28) in the form

$$\partial W_{3R}(H) = \underset{\vartheta_3=0}{\overset{2\pi}{\mathrm{env}}} T_{R_1 R_2}(H) \tag{3.1.29}$$

where "env" is an envelope operator performing an envelope process.

Hence, the boundary $\partial W_{(N\text{-}j+1)R}$ (H) of a (N-j+1)R hyper-ring

can be described as an envelope of the torus family traced by all the points on $\partial W_{(N\text{-}j)R}$ (H), and it can be expressed as

$$\partial W_{(N-j+1)R}(H) = \underset{\vartheta_j,\vartheta_{j+1}=0}{\overset{2\pi}{\mathrm{env}}} T_{R_j R_{j+1}}\left[\partial W_{(N-j)R}(H)\right] \quad . \tag{3.1.32}$$

Thus, a workspace boundary $\partial W_{NR}(H)$ of a general N-R manipulator can be generated by using recursively Eq.(3.1.32), to determine the tori envelopes from the ring up to the N-R hyper-ring in the chain, by computing from the extremity to the base of the manipulator chain .

Telescopic manipulators with prismatic joints

workspace volume W(H) of **Cylindroid Ring**
can be thought as the union of the points swept by revolving cone $T_{R2P}(H)$, due to the mobility in R_2 and P joints, during the θ_1 revolution about Z_1 axis.

$$W(H) = \bigcup_{\vartheta_1=0}^{2\pi} T_{R_2P}(H) \quad . \tag{3.1.33}$$

Alternatively, a Cylindroid Ring $W_{RRP}(H)$ can be considered as the union of the tori $T_{R1R2}(H)$, which are due to the mobility in R_1 and R_2 joints and are traced by all parallel circles which can be cut on the generating cylindrical cone $T_{R2P}(H)$ so that $W_{RRP}(H)$ can be expressed in the form

$$W_{RRP}(H) = \bigcup_{d=d_{min}}^{d_{max}} T_{R_1R_2}(H) \quad . \tag{3.1.34}$$

Consequently, the workspace boundary $\partial W_{RRP}(H)$ of a Cylindroid Ring can be obtained as the envelope of toroidal surfaces $T_{R1R2}(H)$ as

$$\partial W_{RRP}(H) = \underset{d=d_{min}}{\overset{d_{max}}{\mathrm{env}}}\, T_{R_1R_2}(H) \tag{3.1.35}$$

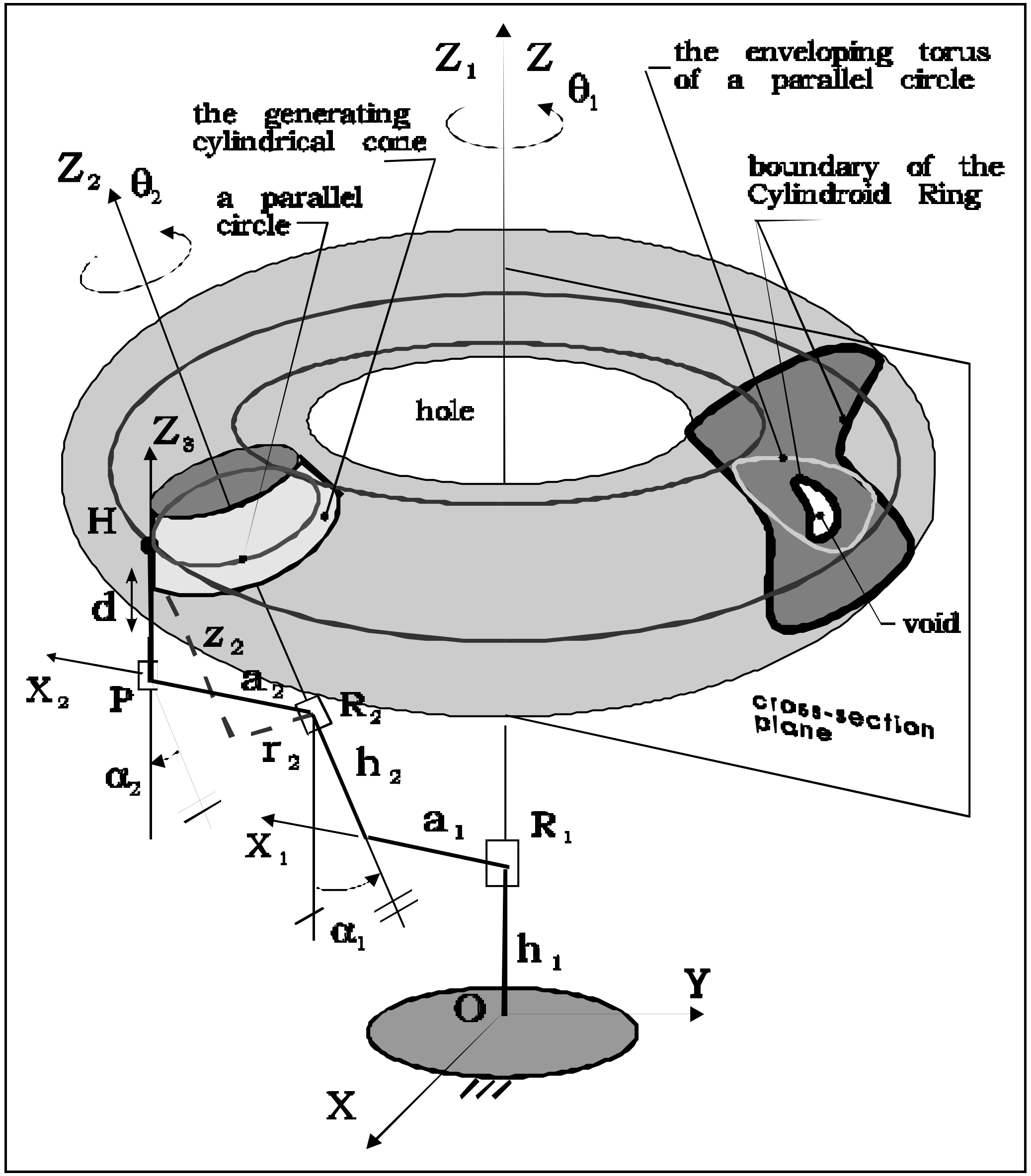


Fig.3.12: Design parameters and topology in the generation of so-called Cylindroid Ring workspace for

telescopic manipulator arms.

A hole is a region, outside the ring but surrounded by the ring yet, within which it is possible to individuate at least a straight line of points not belonging to the ring.

- **Therefore, a hole is generated, when the revolving torus does not intersect or touch the revolution axis.**

A void can be generally identified as an internal region, within the workspace itself, which is not reachable by point H.

- **a so-called ring void** is a void with ring topology, which is generated by a hole in a generating torus that does not intersect the revolution axis.

- **A so-called apple void** is obtained when the hole of the generating torus intersects the axis of revolution for ring generation and a bulk apple shaped volume characterizes it.

Particularly, referring to the case of position workspace of manipulators with revolute joins, two branches of envelope boundary contours in a cross-section of a workspace boundary are observable: an external one and an internal one, as shown in the example of Fig.3.13 for 4R manipulators.

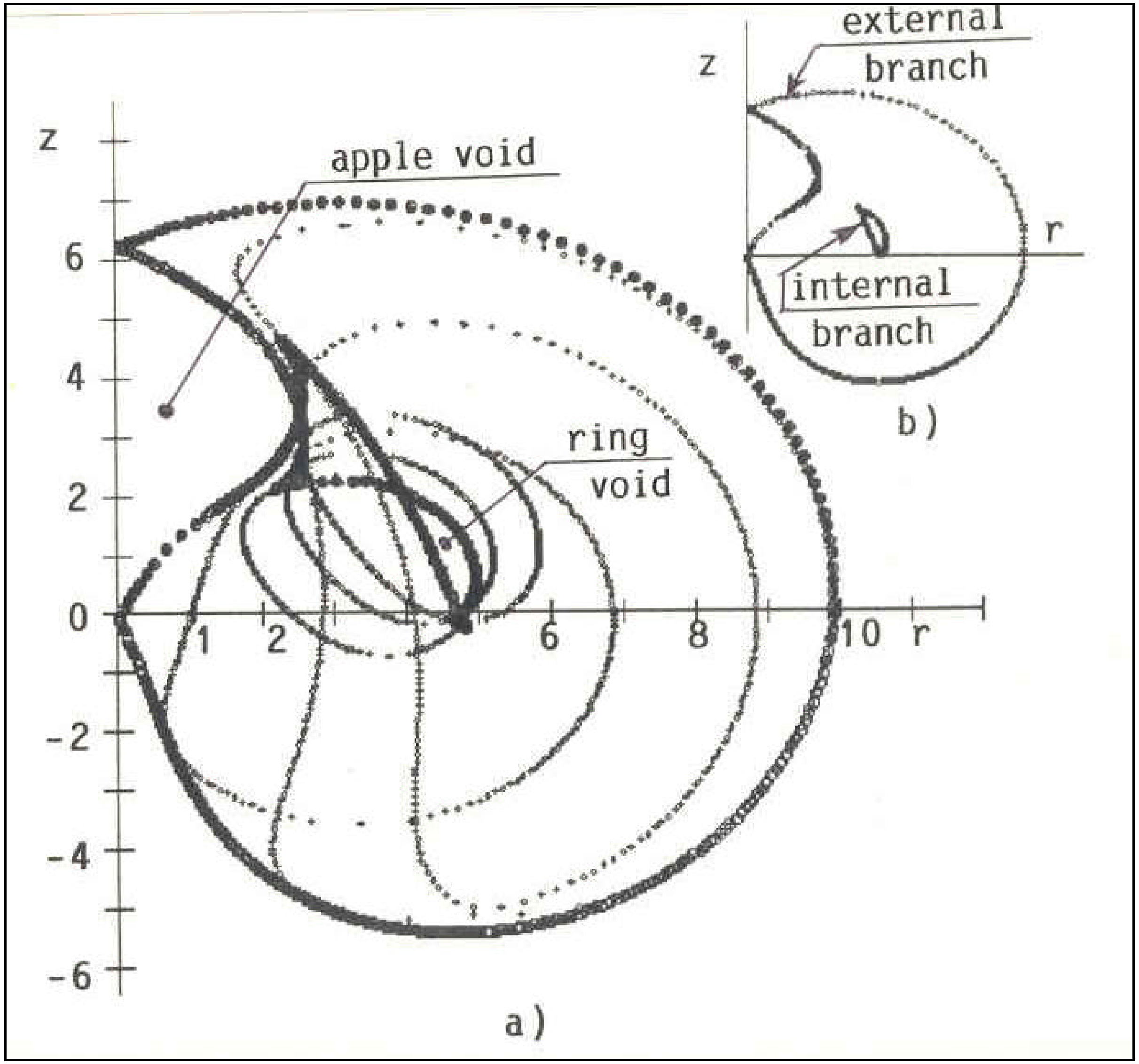


Fig.3.13: Workspace cross-section of a 4R manipulator with a1= 3 u, a2= 4 u, a3= 2 u, a4= 5 u, a1= a2= a3= 60 deg, d2= d3= d4= 1 u. (u is the unit length)

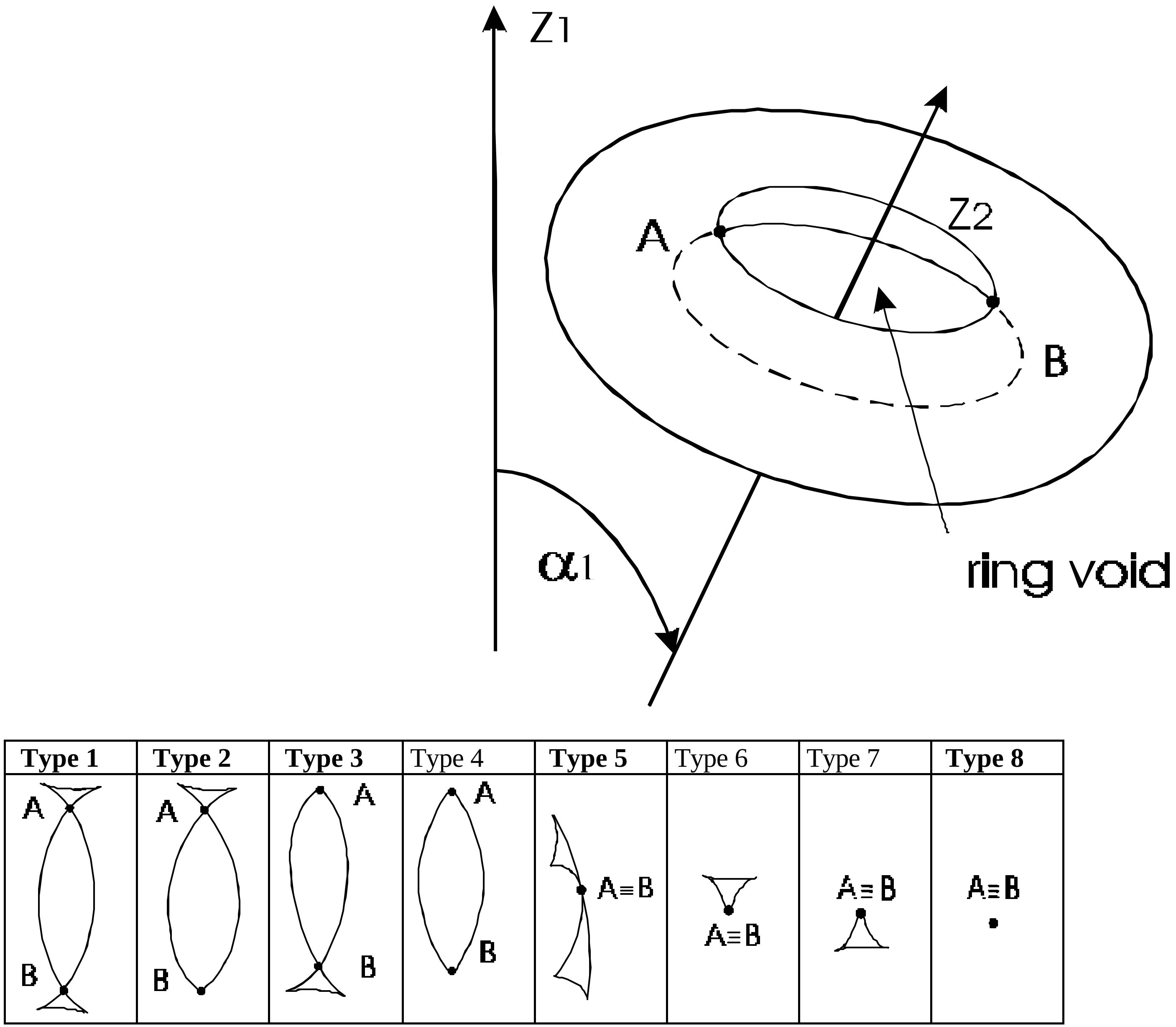


Figure 3. Manifolds for ring void of three-revolute manipulator.

Numerical procedures for workspace determination

- **A binary matrix formulation**
- **An algebraic formulation**

binary matrix formulation - a numerical algorithm whose basic steps are:

1. Dividing the cross-section plane r, z into I x J small rectangles of width Δi and of height Δj, where J and I are the number of divisions along the r axis and z axis, respectively.

Each rectangle is individuated by P_{ij} to provide a binary image of the workspace cross-section.

the width Δi and of the height Δj can be properly selected as a function of the $\Delta\theta_k$ scanning intervals.

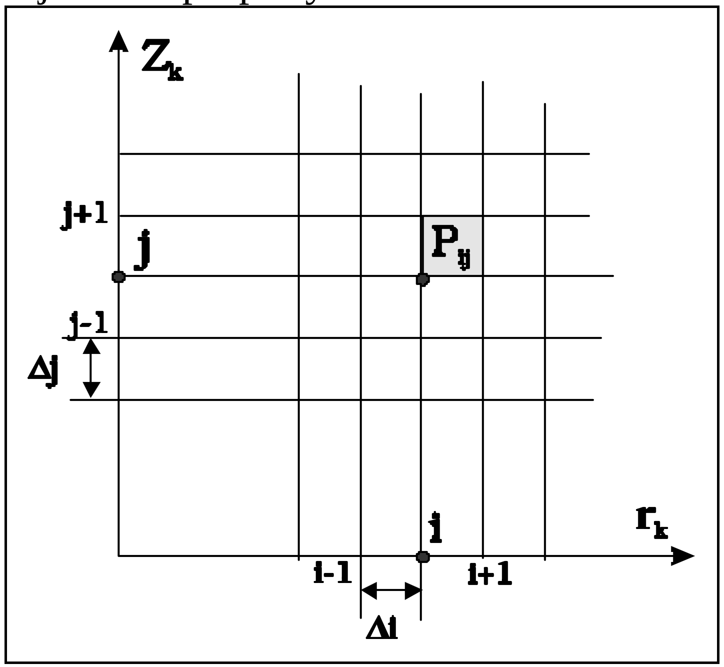


Fig.3.14: A grid of scanning process for binary mapping of workspace cross-section.

2. Initialization by setting P_{ij}=0 for all i and j.

3. A scanning process for each joint angle θ_k from θ_{kmin} up to θ_{kmax} with step $\Delta\theta_k$ to compute workspace point coordinates by using a matrix approach in the form

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_k = {}_k T_{k+1} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{k+1} . \qquad (3.1.37)$$

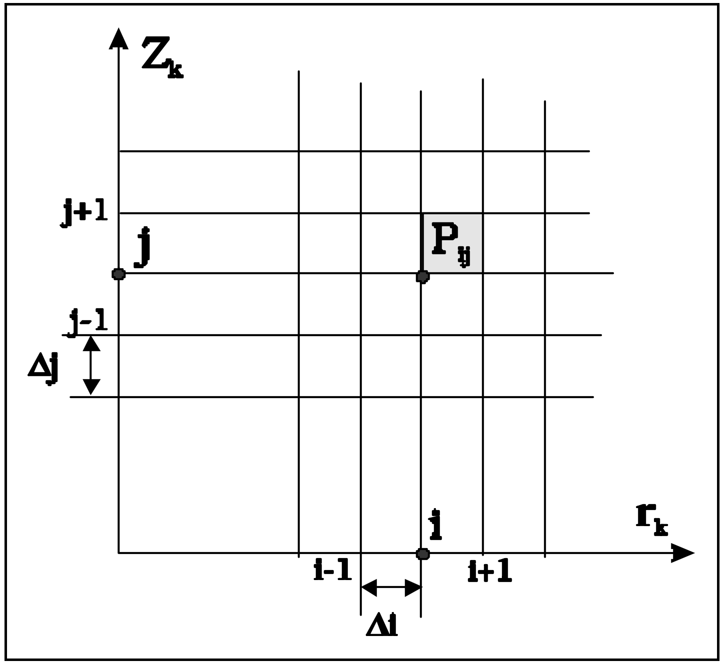
4. Construction of the binary map P_{ij}= 1 of the workspace cross-section by determining i and j as

$$i = \text{fix}\left[\frac{r_k}{\Delta i}\right] \qquad j = \text{fix}\left[\frac{z_k}{\Delta j}\right] \qquad (3.1.38)$$

by using the operator fix to compute the integer value of the above-mentioned ratios.

Therefore, the binary mapping for the workspace cross-section is given by

$$P_{ij} = \begin{cases} 0 \ \text{if} \ \ P_{ij} \notin W(H) \\ 1 \ \text{if} \ \ P_{ij} \in W(H) \end{cases} \tag{3.1.39}$$

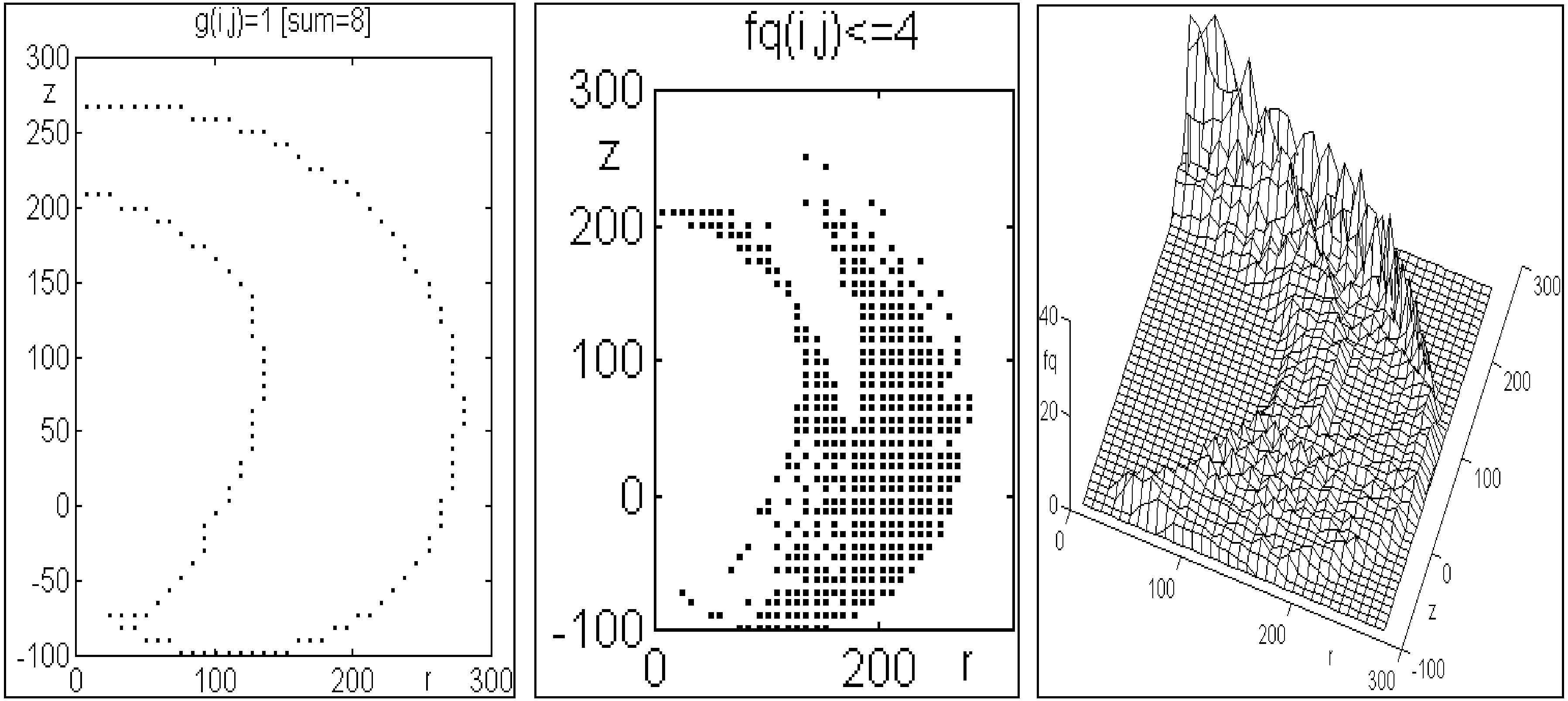


determining workspace boundary can be developed by using P_{ij} yet.

$$\text{sum} = \sum_{i-1}^{i+1} \sum_{j-1}^{j+1} P_{ij} \leq 9 \tag{3.1.40}$$

whose detection can be used to generate a binary mapping $\mathbf{G_{ij} = 1}$ for the boundary points.

- a frequency matrix with entries fq_{ij} can be generated during the generation of P_{ij} itself by giving to fq_{ij} the values of the number of times that P_{ij} has been reached



a) b) c)

Fig.3.15: Results of the binary matrix formulation for workspace evaluation of a robot manipulator COMAU SMART 6.100A with a_1 = 300 mm, α_1 = 90 deg., a_2 = 1,100 mm, $\alpha_2 = \alpha_3$ = 0 deg., a_3 = 1,625 mm, and $b_1 = b_2 = b_3 = 0$: a) cross-section workspace contour generated with sum = 8; b) cross-section area of secondary workspace with $fq(i,j) \leq 4$; c) a map for the frequency matrix $fq(i,j)$.

An algebraic formulation

The boundary of an (N-j+1)R hyper-ring can be expressed algebraically when it is thought generated by enveloping the torus family traced by the parallel circles in the boundary of the revolving (N-j)R hyper-ring, according to Eq.(3.1.32).

$$\partial W_{3R}(H) = \operatorname*{env}_{\vartheta_3=0}^{2\pi} T_{R_1R_2}(H) \qquad (3.1.29)$$

An equation for a torus family can be expressed with respect to the j-th link frame, assuming $C_j \neq 0$ and $\cos\alpha_j \neq 0$, as a function of the radial r_j and axial z_j reaches, in the form

$$(r_j^2 + z_j^2 - A_j)^2 + (C_j z_j + D_j)^2 + B_j = 0 \qquad (3.1.41)$$

where the **so-called torus parameters** are a_j, α_j, r_{j+1}, z_{j+1}, and the coefficients are given as

$$A_j = a_j^2 + r_{j+1}^2 + (z_{j+1} + d_{j+1})^2 \qquad B_j = -4a_j^2 r_{j+1}^2 \qquad (3.1.42)$$

$$C_j = 2a_j / s\alpha_j \qquad D_j = -2a_j\left(z_{j+1} + d_{j+1}\right)c\alpha_j / s\alpha_j$$

- Particular cases with $C_j = 0$ or $\cos\alpha_j = 0$ are not represented by Eq.(3.1.41) and specific formulation can be developed when the torus boundary is not generated as an envelope of the revolving circle.

- The envelope equations of a torus family can be obtained from Eq.(3.1.41) and its derivative with respect to the torus family parameter θ_N.

After some algebra for which $C_j \neq 0$ and $E_j \neq 0$ are needed,
the so-called ring boundary equations can be obtained in the form

$$r_j = \left[A_j - z_j^2 + \frac{\left(C_j z_j + D_j\right) G_j + F_j}{E_j} \right]^{1/2}$$

$$z_j = \frac{-F_j G_j \pm Q_j^{1/2}}{\left(E_j^2 + G_j^2\right) C_j} - \frac{D_j}{C_j}$$

where the **so-called ring coefficients** are given as

$$E_j = R_{j+1} + S_{j+1} \qquad F_j = -2 a_j^2 R_{j+1} \qquad (3.1.44)$$

$$G_j = -2 a_j z'_{j+1} c\alpha_j / s\alpha_j \qquad Q_j = -E_j^2 \left[F_j^2 + B_j \left(E_j^2 + G_j^2 \right) \right]$$

with

$$R_{j+1} = \left[(C_{j+1} z_{j+1} + D_{j+1})(E_{j+1} G'_{j+1} - G_{j+1} E'_{j+1}) + F'_{j+1} E_{j+1} - F_{j+1} E'_{j+1} \right] / E^2_{j+1}$$
$$+ G_{j+1} E_{j+1} (C_{j+1} z'_{j+1} + G_{j+1}) / E^2_{j+1} + E_{j+1} - 2 z_{j+1} z'_{j+1} \quad ,$$

$$S_{j+1} = 2\left(z_{j+1} + d_{j+1} \right) z'_{j+1} \qquad (3.1.45)$$

$$z'_{j+1} = \left[\left(\pm 0.5\, Q'_{j+1} Q^{-1/2}_{j+1} - F_{j+1} G'_{j+1} - G_{j+1} F'_{j+1} \right) \left(E^2_{j+1} + G^2_{j+1} \right) \right.$$
$$\left. + 2\left(F_{j+1} G_{j+1} \mp Q^{1/2}_{j+1} \right) \left(E_{j+1} E'_{j+1} + G_{j+1} G'_{j+1} \right) \right] / \left(E^2_{j+1} + G^2_{j+1} \right)^2 C_{j+1} - G_{j+1} / C_{j+1}$$

The symbol ' represents the derivative operator with respect to the torus family parameter $\boldsymbol{q}_N$ *.*

$$E'_j = R'_{j+1} + S'_{j+1} \qquad F'_j = -2 a^2_j R'_{j+1} \qquad G'_j = -2 a_j\, z''_{j+1}\, c\alpha_j / s\alpha_j \qquad (3.1.46)$$
$$Q'_j = -2 E^2_j \left[F_j \left(F'_j + E^2_j + G^2_j \right) + B_j (E_j E'_j + G_j G'_j) \right] + 2 Q_j E'_j / E_j$$

This iterative computation can be expressed, according to Eqs.(3.1.46), in a general iterative form

$$E^k_{j+1} = R^k_{j+2} + S^k_{j+2} \qquad G^k_{j+1} = -2a_{j+1}\, z^{k+1}_{j+2}\, c\alpha_{j+1}/s\alpha_{j+1}$$

$$F^k_{j+1} = -2a^2_{j+1} R^k_{j+2} \qquad k = 0, 1,..., j;\ j = 0,1,..., N\text{-}4 \qquad (3.1.47)$$

and, according to Eqs.(3.1.45), as

$$R^k_{j+2} = f^k\left(E^{k+1}_{j+2}, F^{k+1}_{j+2}, G^{k+1}_{j+2}\right) \qquad z^{k+1}_{j+2} = h^k\left(E^{k+1}_{j+2}, F^{k+1}_{j+2}, G^{k+1}_{j+2}\right)$$

$$S^k_{j+2} = g^k\left(E^{k+1}_{j+2}, F^{k+1}_{j+2}, G^{k+1}_{j+2}\right) \qquad k = 0, 1,..., j;\ j = 0,1,..., N\text{-}4 \qquad (3.1.48)$$

the ring coefficients can be algebraically expressed from the ring equations as

$$E_{N-2} = -2a_N\,(d_{N-1}\, s\alpha_{N-1} c\theta_N + a_{N-1}\, s\,\theta_N)$$

$$F_{N-2} = 4a^2_{N-2} a_N\,(a_N\, s^2\alpha_{N-1} s\,\theta_N c\theta_N + a_{N-1} s\theta_N - d_N s\alpha_{N-1} c\alpha_{N-1} c\theta_N)$$

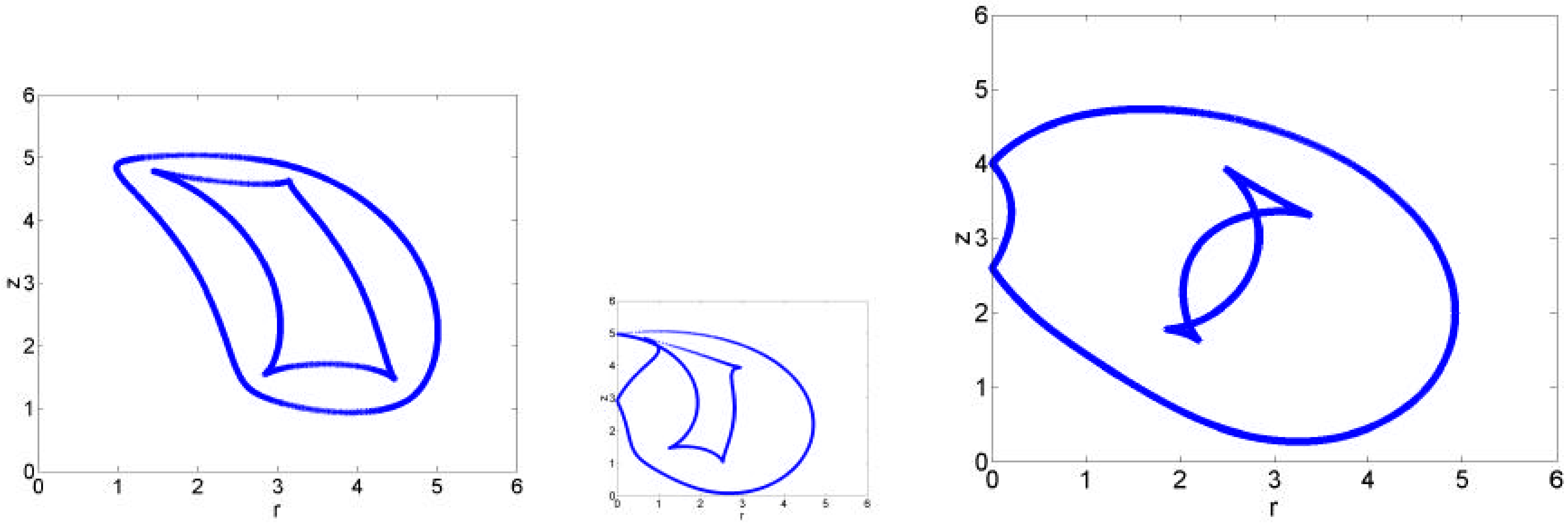
$$G_{N-2} = 2a_{N-2} a_N\, c\alpha_{N-2} s\alpha_{N-1} c\theta_N / s\alpha_{N-2} \qquad (3.1.49)$$

where from the geometry of the manipulator chain it holds

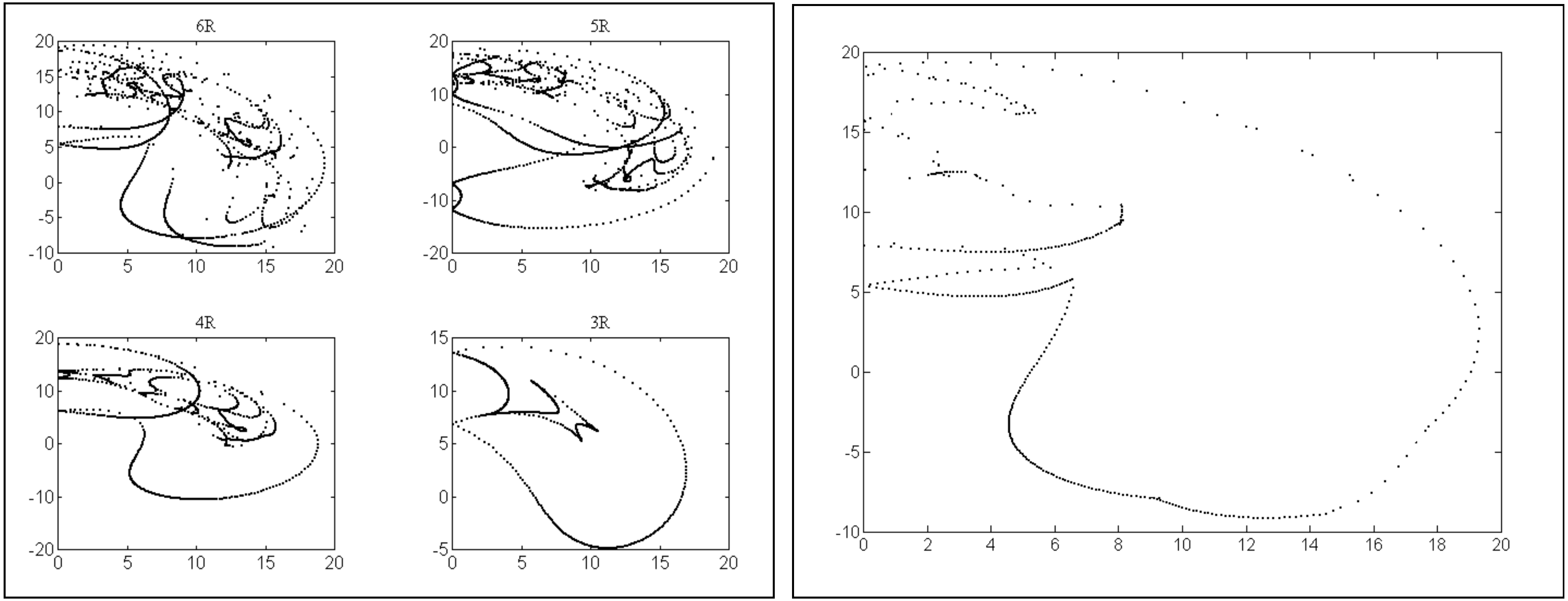
$$r_{N-1} = \left[(a_N c\theta_N + a_{N-1})^2 + (a_N s\,\theta_N c\alpha_{N-1} + d_N s\alpha_{N-1})^2\right]^{1/2}$$

$$z_{N-1} = d_N\, c\alpha_{N-1} - a_N\, s\theta_N\, s\alpha_{N-1} \qquad (3.1.50)$$

- This can be obtained by scanning the joint angle θ_N from 0 to 2π and calculating at each j the coefficients A_j, B_j, C_j, D_j, E_j, F_j, G_j, R_j, S_j and z'_j, and finally r_j ,z_j when the j derivatives of E_j, F_j, G_j, are evaluated by using previous calculations for R_{j+1}, S_{j+1} and z'_{j+1}.
-



examples of ring workspace of 3R manipulators



a) b)

Fig.3.16: Workspace cross-sections of a 6R manipulator with a_i= i u (i = 1,...,6), d_1 = 0 u, d_i = (i-1) u (i = 2,...,5), and $\alpha_i = \pi/4$ (i = 1,...,5). *(u is the unit length): a) the envelope boundary in the generating workspaces; b) the cross-section of primary workspace.*

- **A similar algebraic formulation can be deduced for the workspace of manipulators with prismatic joints.**

Figure 3.12 shows that the workspace boundary of Cylindroid Ring is composed of two different geometrical topologies: envelope segments and toroidal surfaces.

The envelope segments are located in the lateral sides of the cross-section representation, and two toroidal surfaces are the top and bottom covers, respectively.

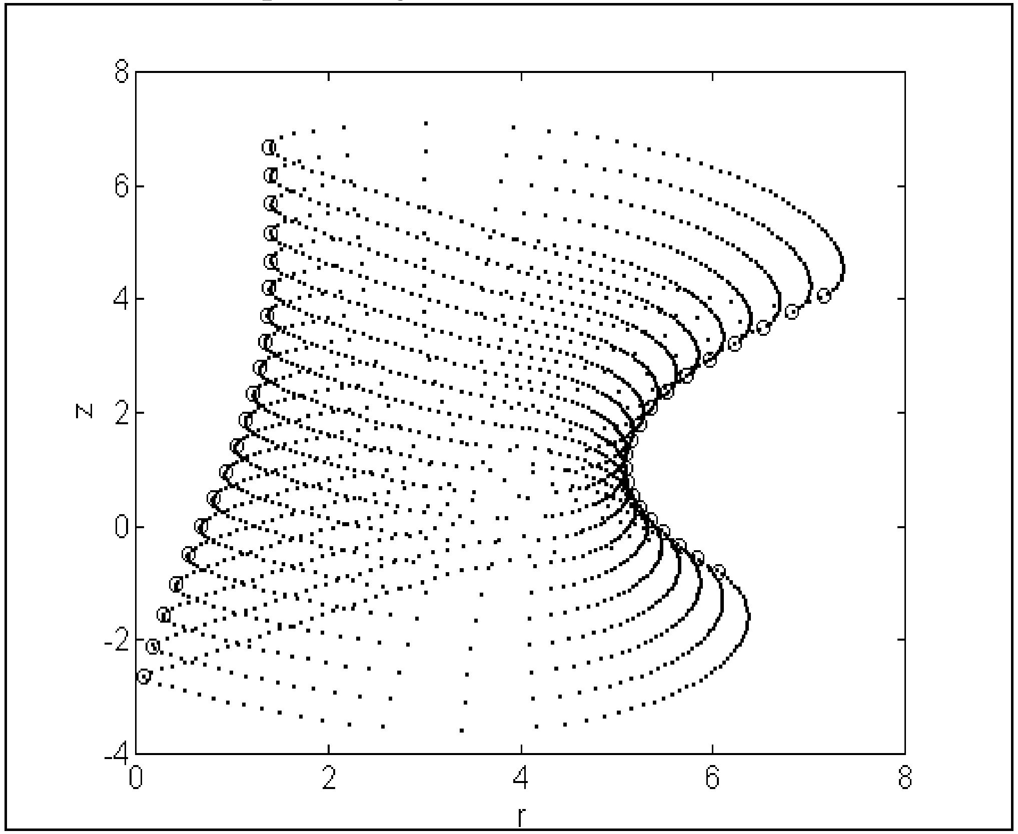


Figure 3.17: Cross-section of a workspace boundary computed as tori envelope segments (circle points) and toroidal covers (dotted points) for a telescopic manipulator arm with $a_1 = 3$ u, $a_2 = 2$ u, $h_1 = 0$ u, $h_2 = 2$ u, $\alpha_1 = 30$ deg, $\alpha_2 = 30$ deg., $d_{min} = -5$ u, $d_{max} = 5$ u, (u is a unit length).

Torus covers

$$\left(r^2 + z^2 - A\right)^2 + \left(C\,z + D\right)^2 + B = 0 \qquad (3.1.51)$$

The so-called structural coefficients are expressed as

$$A = a_1^2 + r_2^2 + \left(z_2 + h_2\right)^2 \qquad B = -4\,a_1^2\,r_2^2$$

$$C = 2\,\frac{a_1}{\sin\alpha_1} \qquad D = -2\,a_1\left(z_2 + h_2\right)\frac{\cos\alpha_1}{\sin\alpha_1} \qquad (3.1.52)$$

$$r_2 = \sqrt{a_2^2 + d^2\sin^2\alpha_2} \qquad z_2 = d\cos\alpha_2 \qquad (3.1.53)$$

where the independent variable is the stroke parameter d.

The envelope segments

the workspace boundary $\partial W(H)$ of a Cylindroid Ring can be obtained as the envelope of toroidal surfaces $T_{R1R2}(H)$, which are generated by revolution of the parallel circles of the generating cylindrical cone, and straight-line segments of the envelope contour.

$$z = \frac{-B'D' \pm A'\sqrt{-\left[B'^2 + B\left(A'^2 + D'^2\right)\right]}}{\left(A'^2 + D'^2\right)C} - \frac{D}{C}$$

(3.1.57)

$$r = \sqrt{\frac{B' + (C\,z + D)\,D'}{A'} + A - z^2}$$

$$A' = 2\left(d + h_2 \cos\alpha_2\right) \qquad B' = -4\,a_1^2\,d \sin^2\alpha_2 \qquad (3.1.56)$$

$$D' = -2\,a_1 \cos\alpha_2 \frac{\cos\alpha_1}{\sin\alpha_1}$$

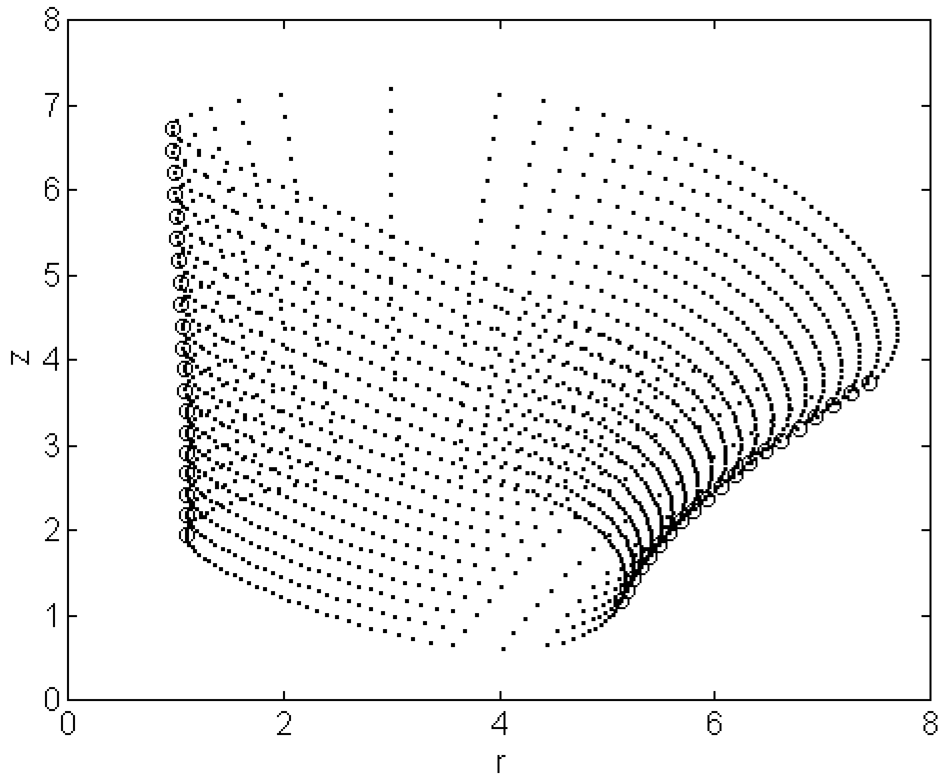


Fig.2. Cross-section of a Cylindroid Ring workspace volume computed as an union of generating tori for a telescopic RRP manipulator with a_1 = 3 u, a_2 = 2 u, d_1 = 0 u, d_2 = 1 u, α_1 = 30 deg, a2 = 30 deg., d_{min} = 1 u, d_{max} = 6 u, (u is a unit length).

Table 1 - Effect of twist angles α_1 and α_2 on the cross-section shape of workspace for telescopic RRP manipulators with a_1 = 1 u, a_2 = 2 u, d_1 = 0 u, d_2 = 1 u, d_{min} = - 5 u, d_{max} = 5 u, (u is a unit length). (see Fig.5).