Analysis and Synthesis of Manipulator Workspace

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1. Kinematic chain and H-D notation for manipulators

Fig. 3.1: A scheme for the manipulator architecture of a robot with the arm, wrist, and end-effector.

Fig. 3.2: Planar examples of kinematic chains of manipulators: a) serial chain as open type; b) parallel chain as closed type.
Fig. 3.4: Manipulator architectures for industrial robots.
A kinematic model of a manipulator can be named as functional when its scheme refers to kinematic parameters only, but permits also to understand the motion capability of the manipulator architecture.

A kinematic functional model can be determined from the mechanical design of a robot through the following step-by-step procedure:
- identification of the type of the joints;
- identification of the position of each joint axes;
- identification of the geometry of the links;
- drawing of a scheme for the kinematic chain.

![Fig.3.5: An example of modeling an industrial robot: a) the mechanical design; b) the corresponding kinematic functional scheme.](image)
Fig. 3.6: A kinematic scheme for manipulator link parameters according to the H-D notation.
The kinematic parameters of a manipulator can be defined according to the H-D notation in Fig.3.6 as:
- $a_j$, which is named as the link length that is measured as the distance between the $Z_j$ and $Z_{j+1}$ axes along $X_j$;
- $\alpha_j$, which is named as the twist angle that is measured as the angle between the $Z_j$ and $Z_{j+1}$ axes about $X_j$;
- $d_{j+1}$, which is named as the link offset that is measured as the distance between $X_j$ and $X_{j+1}$ axes along $Z_{j+1}$;
- $\theta_{j+1}$, which is named as the joint angle that is measured as the angle between the $X_j$ and $X_{j+1}$ axes about $Z_{j+1}$.

When a joint can be modeled as a rotation pair, the angle $\theta_{j+1}$ is the corresponding kinematic variable. When a joint can be modeled as a prismatic pair, the distance $d_{j+1}$ is the corresponding kinematic variable.
the position problem can be considered from different viewpoints depending of the unknowns:

- **Kinematic Direct Problem** in which the dimensions of a manipulator are given through the dimensional H-D parameters of the links but the position and orientation of the end-effector are determined as a function of the values of the joint variables;

- **Kinematic Inverse problem** in which the position and orientation of the end-effector of a given manipulator are given, and the configuration of the manipulator chain is determined by computing the values of the joint values.

- **Kinematic Indirect Problem** (properly Design Problem) in which a certain number of positions and orientations of the end-effector are given but the type of manipulator chain and its dimensions are the unknowns of the problem.
Transformation matrix

Fig. 3.7: Vectors for relative position and orientation between two reference frames.

\[
X = h + Rx
\]

\[
X = T x
\]

When the Transformation Matrix is defined as

\[
T = \begin{bmatrix}
R & h \\
0 & 1
\end{bmatrix}
\]

Rotation Matrix:

\[
R = \begin{bmatrix}
i \cdot I & i \cdot J & i \cdot K \\
j \cdot I & j \cdot J & j \cdot K \\
k \cdot I & k \cdot J & k \cdot K
\end{bmatrix}
\]
\[ \mathbf{X}_0 = 0T_N \mathbf{X}_N \]

and therefore a resultant transformation matrix for a manipulator can be obtained as

\[ 0T_N = 0T_1 T_2 \ldots N-1T_N = \prod_{k=0}^{N-1} T_{k+1} \]

This expression can be considered the fundamental typical formulation for the Direct Kinematic of manipulators.

Fig. 3.8: A scheme for indicating which frames a transformation matrix is formulated with respect to.
In particular, by using the above-mentioned properties and formulation for the transformation matrix, a general expression for $kT_{k+1}$ can be given by referring to the general manipulator scheme of Fig. 3.6, in the form

$$
k T_{k+1} = \begin{bmatrix}
\cos \theta_{k+1} & -\sin \theta_{k+1} & 0 & a_k \\
\sin \theta_{k+1} \cos \alpha_k & \cos \theta_{k+1} \cos \alpha_k & -\sin \alpha_k & -\sin \alpha_k \ d_{k+1} \\
\sin \theta_{k+1} \sin \alpha_k & \cos \theta_{k+1} \sin \alpha_k & \cos \alpha_k & \cos \alpha_k \ d_{k+1} \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

as a composition of elementary matrices that describe elementary differences between reference frames that are described by the H-D parameters. An elementary matrix corresponds to a rotation about a reference axis or a translation along such an axis.

Thus the matrix $kT_{k+1}$ can be given by the expression

$$k T_{k+1} = \text{Rot}(X_k, \alpha_k) \ \text{Trasl}(X_k, a_k) \ \text{Rot}(Z_{k+1}, \theta_{k+1}) \ \text{Trasl}(Z_{k+1}, d_{k+1}) \quad (3.1.19)$$

or alternatively by using elementary helicoidal motions, it is given by

$$k T_{k+1} = \text{Screw}(X_k, a_k, \alpha_k) \ \text{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1}) \quad (3.1.20)$$
Definition and Generation of workspace

- **The workspace** $W(H)$ is defined as the region of points that can be reached by a reference point $H$ on the manipulator extremity. This is the Position Workspace.

The workspace is defined as generated by a reference point $H$ on the extremity of the manipulator chain that is moved to reach all possible positions because of mobility ranges of the joints.

- **Similarly, the Orientation Workspace can be defined as the set of orientations that can be reached by the manipulator extremity.**

Fundamental characteristics of the manipulator position workspace are recognized as:

- **the shape and volume** of the workspace, which is a solid of revolution for manipulators with only revolute joints, and can be a parallelepiped for manipulators with prismatic joints;
- **the hole**, whose existence is determined by a region of unreachable points that can individuate straight-lines surrounded by the workspace yet;
- **the voids**, which are regions of unreachable points that are buried within the workspace yet.

Similar characteristics can be identified for the orientation workspace that nevertheless has a different topology.
Revolving a torus about an axis generates a ring:

\[ W_{3R}(H) = \bigcup_{\vartheta_1=0}^{2\pi} T_{R_2R_3}(H) \]  \hspace{1cm} (3.1.27)

Alternatively, a ring \( W_{3R}(H) \) can be considered as the union of the tori \( T_{R_1R_2}(H) \),

\[ W_{3R}(H) = \bigcup_{\vartheta_3=0}^{2\pi} T_{R_1R_2}(H) \]  \hspace{1cm} (3.1.28)
Fig. 3.11: A descriptive view of workspace generation for serial manipulators with revolute joints.
The generation process of a hyper-ring
is a consecutive revolving process of a circle, a torus, a ring, a 4R hyper-ring, and so on.
This can be expressed through a revolution operator Rev in the form

$$\text{Rev}_v W_{(N-j)R}(H)$$

for

$$j = 1, \ldots, N-1$$

(3.1.30)

Alternatively,
$W_{(N-j+1)R}(H)$ can be considered as the union of a suitable torus family which is traced by
the boundary points in the revolving torus, ring, 4R hyper-ring and so on, when they are rotated
completely about the first two revolute axes in the corresponding generating sub-chain. It can be expressed in the form

$$W_{(N+1-j)R}(H) = \bigcup_{\theta_j, \theta_{j+1}=0}^{2\pi} T_{R_j R_{j+1}} \left[ \partial W_{(N-j)R}(H) \right]$$

(3.1.31)

where $T_{R_j R_{j+1}}(H)$ represents a torus generated by revolutions $\theta_j$ and $\theta_{j+1}$ about the joints axes of $R_j$ and $R_{j+1}$. The revolution in $\theta_{j+1}$ generates a parallel circle in $W_{(N-j)R}(H)$ and together with $\theta_j$ generates the torus $T_{R_j R_{j+1}}(H)$. 
the boundary $\partial W_{3R}(H)$ of a ring can be thought as the envelope of torus surfaces generated by revolution of the generating torus or, alternatively, it can be obtained by an envelope of torus surfaces that are traced from the parallel circles of the generating torus. The latter procedure can be expressed according to Eq.(3.1.28) in the form

$$
\partial W_{3R}(H) = \text{env} T_{R_1R_2}(H)
$$

where "env" is an envelope operator performing an envelope process.

Hence, the boundary $\partial W_{(N-j+1)R}(H)$ of a $(N-j+1)R$ hyper-ring can be described as an envelope of the torus family traced by all the points on $\partial W_{(N-j)R}(H)$, and it can be expressed as

$$
\partial W_{(N-j+1)R}(H) = \text{env}_{\theta_j, \theta_{j+1}=0} T_{R_jR_{j+1}}[\partial W_{(N-j)R}(H)].
$$

(3.1.32)

Thus, a workspace boundary $\partial W_{NR}(H)$ of a general N-R manipulator can be generated by using recursively Eq.(3.1.32), to determine the tori envelopes from the ring up to the N-R hyper-ring in the chain, by computing from the extremity to the base of the manipulator chain.
Telescopic manipulators with prismatic joints

workspace volume $W(H)$ of **Cylindroid Ring**
can be thought as the union of the points swept by revolving cone $T_{R_2P}(H)$, due to the mobility in $R_2$ and $P$ joints, during the $\theta_1$ revolution about $Z_1$ axis.

$$W(H) = \bigcup_{\theta_1=0}^{2\pi} T_{R_2P}(H) .$$  \hspace{1cm} (3.1.33)

**Alternatively**, a Cylindroid Ring $W_{RRP}(H)$ can be considered as the union of the tori $T_{R_1R_2}(H)$, which are due to the mobility in $R_1$ and $R_2$ joints and are traced by all parallel circles which can be cut on the generating cylindrical cone $T_{R_2P}(H)$ so that $W_{RRP}(H)$ can be expressed in the form

$$W_{RRP}(H) = \bigcup_{d=d_{\text{min}}}^{d_{\text{max}}} T_{R_1R_2}(H) .$$  \hspace{1cm} (3.1.34)

Consequently, the workspace boundary $\partial W_{RRP}(H)$ of a Cylindroid Ring can be obtained as the envelope of toroidal surfaces $T_{R_1R_2}(H)$ as

$$\partial W_{RRP}(H) = \text{env} \bigcup_{d=d_{\text{min}}}^{d_{\text{max}}} T_{R_1R_2}(H) .$$  \hspace{1cm} (3.1.35)
Fig.3.12: Design parameters and topology in the generation of so-called Cylindroid Ring workspace for
telescopic manipulator arms.

A hole is a region, outside the ring but surrounded by the ring yet, within which it is possible to individuate at least a straight line of points not belonging to the ring.

- Therefore, a hole is generated, when the revolving torus does not intersect or touch the revolution axis.

A void can be generally identified as an internal region, within the workspace itself, which is not reachable by point H.

- a so-called ring void is a void with ring topology, which is generated by a hole in a generating torus that does not intersect the revolution axis.

- A so-called apple void is obtained when the hole of the generating torus intersects the axis of revolution for ring generation and a bulk apple shaped volume characterizes it.

Particularly, referring to the case of position workspace of manipulators with revolute joins, two branches of envelope boundary contours in a cross-section of a workspace boundary are observable: an external one and an internal one, as shown in the example of Fig.3.13 for 4R manipulators.
Fig. 3.13: Workspace cross-section of a 4R manipulator with $a_1 = 3\, u$, $a_2 = 4\, u$, $a_3 = 2\, u$, $a_4 = 5\, u$, $a_1 = a_2 = a_3 = 60$ deg, $d_2 = d_3 = d_4 = 1\, u$. ($u$ is the unit length)
Figure 3. Manifolds for ring void of three-revolute manipulator.
Numerical procedures for workspace determination

- A binary matrix formulation
- An algebraic formulation

binary matrix formulation - a numerical algorithm whose basic steps are:
1. Dividing the cross-section plane r, z into I x J small rectangles of width $\Delta i$ and of height $\Delta j$, where J and I are the number of divisions along the r axis and z axis, respectively.

Each rectangle is individuated by $P_{ij}$ to provide a binary image of the workspace cross-section.

the width $\Delta i$ and of the height $\Delta j$ can be properly selected as a function of the $\Delta \theta_k$ scanning intervals.

Fig.3.14: A grid of scanning process for binary mapping of workspace cross-section.
2. Initialization by setting $P_{ij}=0$ for all $i$ and $j$.
3. A scanning process for each joint angle $\theta_k$ from $\theta_{k\min}$ up to $\theta_{k\max}$ with step $\Delta \theta_k$ to compute workspace point coordinates by using a matrix approach in the form

$$
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}_k = T_{k+1} \begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}_{k+1}.
$$

(3.1.37)

4. Construction of the binary map $P_{ij}=1$ of the workspace cross-section by determining $i$ and $j$ as

$$
i = \text{fix} \left[ \frac{r_k}{\Delta i} \right] \quad \text{and} \quad j = \text{fix} \left[ \frac{z_k}{\Delta j} \right]
$$

(3.1.38)

by using the operator \text{fix} to compute the integer value of the above-mentioned ratios.
Therefore, the binary mapping for the workspace cross-section is given by

\[
P_{ij} = \begin{cases} 
0 & \text{if } P_{ij} \notin W(H) \\
1 & \text{if } P_{ij} \in W(H) 
\end{cases}
\]  

(3.1.39)

determining workspace boundary can be developed by using \( P_{ij} \) yet.

\[
\sum_{i-1}^{i+1} \sum_{j-1}^{j+1} P_{ij} \leq 9
\]  

(3.1.40)

whose detection can be used to generate a binary mapping \( G_{ij} = 1 \) for the boundary points.
a frequency matrix with entries $f_{ij}$ can be generated during the generation of $P_{ij}$ itself by giving to $f_{ij}$ the values of the number of times that $P_{ij}$ has been reached.

Fig. 3.15: Results of the binary matrix formulation for workspace evaluation of a robot manipulator COMAU SMART 6.100A with $a_1 = 300$ mm, $\alpha_1 = 90$ deg., $a_2 = 1,100$ mm, $\alpha_2 = \alpha_3 = 0$ deg., $a_3 = 1,625$ mm, and $b_1 = b_2 = b_3 = 0$: a) cross-section workspace contour generated with sum = 8; b) cross-section area of secondary workspace with $f_{q(i,j)} \leq 4$; c) a map for the frequency matrix $f_{q(i,j)}$. 
An algebraic formulation

The boundary of an \((N-j+1)R\) hyper-ring can be expressed algebraically when it is thought generated by enveloping the torus family traced by the parallel circles in the boundary of the revolving \((N-j)R\) hyper-ring, according to Eq.(3.1.32).

\[
\partial W_{3R}(H) = \text{env} T_{R_1R_2}(H) \quad \vartheta_3 = 0
\]

An equation for a torus family can be expressed with respect to the \(j\)-th link frame, assuming \(C_j \neq 0\) and \(\cos \alpha_j \neq 0\), as a function of the radial \(r_j\) and axial \(z_j\) reaches, in the form

\[
(r_j^2 + z_j^2 - A_j)^2 + (C_j z_j + D_j)^2 + B_j = 0
\]

where the so-called torus parameters are \(a_j, \alpha_j, r_{j+1}, z_{j+1}\), and the coefficients are given as

\[
A_j = a_j^2 + r_{j+1}^2 + (z_{j+1} + d_{j+1})^2 \quad B_j = -4a_j^2 r_{j+1}^2 \\
C_j = 2a_j / \sin \alpha_j \quad D_j = -2a_j (z_{j+1} + d_{j+1}) \cos \alpha_j / \sin \alpha_j
\]

- Particular cases with \(C_j = 0\) or \(\cos \alpha_j = 0\) are not represented by Eq.(3.1.41) and specific formulation can be developed when the torus boundary is not generated as an envelope of the revolving circle.
• The envelope equations of a torus family can be obtained from Eq.(3.1.41) and its derivative with respect to the torus family parameter $\theta_N$.

After some algebra for which $C_j \neq 0$ and $E_j \neq 0$ are needed, the so-called ring boundary equations can be obtained in the form

\[
\begin{align*}
  r_j &= \left[ A_j - z_j^2 + \frac{(C_j z_j + D_j)G_j + F_j}{E_j} \right]^{1/2} \\
  z_j &= -\frac{F_j G_j \pm Q_j^{1/2}}{(E_j^2 + G_j^2)C_j} \frac{D_j}{C_j}
\end{align*}
\]

where the so-called ring coefficients are given as

\[
\begin{align*}
  E_j &= R_{j+1} + S_{j+1} & F_j &= -2a_j^2 R_{j+1} \\
  G_j &= -2a_j z_{j+1} c\alpha_j / s\alpha_j & Q_j &= -E_j^2 \left| F_j^2 + B_j (E_j^2 + G_j^2) \right|
\end{align*}
\]

with
\[ R_{j+1} = \left( (C_{j+1} z_{j+1} + D_{j+1}) (E_{j+1} G'_{j+1} - G_{j+1} E'_{j+1}) + F'_{j+1} E_{j+1} - F_{j+1} E'_{j+1} \right) / E'_{j+1} \]
\[ + G_{j+1} E_{j+1} (C_{j+1} z_{j+1} + G_{j+1}) / E'_{j+1} + E_{j+1} - 2z_{j+1} z'_{j+1} , \]

\[ S_{j+1} = 2 (z_{j+1} + d_{j+1}) z'_{j+1} \] (3.1.45)

\[ z'_{j+1} = \begin{bmatrix} \pm 0.5 Q'_{j+1} Q^{-1/2} - F_{j+1} G'_{j+1} - G_{j+1} F'_{j+1} \right) \left( E_{j+1}^2 + G_{j+1}^2 \right) \]
\[ + 2 \left( F_{j+1} G_{j+1} + Q^{1/2} \right) \left( E_{j+1} E'_{j+1} + G_{j+1} G'_{j+1} \right) \right) \left( E_{j+1}^2 + G_{j+1}^2 \right) \left( C_{j+1} - G_{j+1} / C_{j+1} \right) \]

The symbol ' represents the derivative operator with respect to the torus family parameter \( \Theta_N \).

\[ E'_{j} = R'_{j+1} + S'_{j+1} \]
\[ F'_{j} = -2a_j^2 R'_{j+1} \]
\[ G'_{j} = -2a_j z''_{j+1} c\alpha_j / s\alpha_j \] (3.1.46)
\[ Q'_{j} = -2E_{j}^2 \left[ F_j \left( F_j + E^2_j + G^2_j \right) + B_j \left( E_j E_j' + G_j G_j' \right) \right] + 2Q_j E_j' / E_j \]
This iterative computation can be expressed, according to Eqs. (3.1.46), in a general iterative form

\[ E_{j+1}^k = R_{j+2}^k + S_{j+2}^k \] \[ G_{j+1}^k = -2a_{j+1} z_{j+2}^{k+1} \frac{c \alpha_{j+1}}{s \alpha_{j+1}} \]

\[ F_{j+1}^k = -2a_j^2 R_{j+2}^k \] \[ k = 0, 1, \ldots, j; j = 0, 1, \ldots, N-4 \quad (3.1.47) \]

and, according to Eqs. (3.1.45), as

\[ R_{j+2}^k = f^k \left( E_{j+2}^{k+1}, F_{j+2}^{k+1}, G_{j+2}^{k+1} \right) \]
\[ z_{j+2}^{k+1} = h^k \left( E_{j+2}^{k+1}, F_{j+2}^{k+1}, G_{j+2}^{k+1} \right) \]
\[ S_{j+2}^k = g^k \left( E_{j+2}^{k+1}, F_{j+2}^{k+1}, G_{j+2}^{k+1} \right) \] \[ k = 0, 1, \ldots, j; j = 0, 1, \ldots, N-4 \quad (3.1.48) \]

The ring coefficients can be algebraically expressed from the ring equations as

\[ E_{N-2} = -2a_N \left( d_{N-1} s \alpha_{N-1} c \theta_N + a_{N-1} s \theta_N \right) \]
\[ F_{N-2} = 4a_{N-2}^2 a_N \left( a_N s^2 \alpha_{N-1} s \theta_N c \theta_N + a_{N-1} s \theta_N - d_N s \alpha_{N-1} c \alpha_{N-1} c \theta_N \right) \]
\[ G_{N-2} = 2a_{N-2} a_N c \alpha_{N-2} s \alpha_{N-1} c \theta_N / s \alpha_{N-2} \quad (3.1.49) \]

where from the geometry of the manipulator chain it holds

\[ r_{N-1} = \left[ \left( a_N c \theta_N + a_{N-1} \right)^2 + \left( a_N s \theta_N c \alpha_{N-1} + d_N s \alpha_{N-1} \right)^2 \right]^{1/2} \]
\[ z_{N-1} = d_N c \alpha_{N-1} - a_N s \theta_N s \alpha_{N-1} \quad (3.1.50) \]
• This can be obtained by scanning the joint angle $\theta_N$ from 0 to $2\pi$ and calculating at each $j$ the coefficients $A_j$, $B_j$, $C_j$, $D_j$, $E_j$, $F_j$, $G_j$, $R_j$, $S_j$ and $z'_j$, and finally $r_j$, $z_j$ when the $j$ derivatives of $E_j$, $F_j$, $G_j$, are evaluated by using previous calculations for $R_{j+1}$, $S_{j+1}$ and $z'_{j+1}$.

• examples of ring workspace of 3R manipulators
Fig. 3.16: Workspace cross-sections of a 6R manipulator with $a_i = i \, u \ (i = 1, \ldots, 6)$, $d_1 = 0 \, u$, $d_i = (i-1) \, u \ (i = 2, \ldots, 5)$, and $\alpha_i = \pi/4 \ (i = 1, \ldots, 5)$. ($u$ is the unit length): a) the envelope boundary in the generating workspaces; b) the cross-section of primary workspace.
• A similar algebraic formulation can be deduced for the workspace of manipulators with prismatic joints.

Figure 3.12 shows that the workspace boundary of Cylindroid Ring is composed of two different geometrical topologies: envelope segments and toroidal surfaces. The envelope segments are located in the lateral sides of the cross-section representation, and two toroidal surfaces are the top and bottom covers, respectively.

Figure 3.17: Cross-section of a workspace boundary computed as tori envelope segments (circle points) and toroidal covers (dotted points) for a telescopic manipulator arm with $a_1 = 3 u$, $a_2 = 2 u$, $h_1 = 0 u$, $h_2 = 2 u$, $\alpha_1 = 30 \text{ deg}$, $\alpha_2 = 30 \text{ deg.}$, $d_{\text{min}} = -5 u$, $d_{\text{max}} = 5 u$, (u is a unit length).
Torus covers

\[
\left( r^2 + z^2 - A \right)^2 + (C z + D)^2 + B = 0 \quad (3.1.51)
\]

The so-called structural coefficients are expressed as

\[
A = a_1^2 + r_2^2 + (z_2 + h_2)^2 \\
B = -4 a_1^2 r_2^2 \\
C = 2 \frac{a_1}{\sin \alpha_1} \\
D = -2 a_1 \left( z_2 + h_2 \right) \frac{\cos \alpha_1}{\sin \alpha_1} \quad (3.1.52)
\]

\[
r_2 = \sqrt{a_2^2 + d^2 \sin^2 \alpha_2} \\
z_2 = d \cos \alpha_2 \quad (3.1.53)
\]

where the independent variable is the stroke parameter \( d \).
The envelope segments
the workspace boundary $\partial W(H)$ of a Cylindroid Ring can be obtained as the envelope of toroidal surfaces $T_{R1R2}(H)$, which are generated by revolution of the parallel circles of the generating cylindrical cone, and straight-line segments of the envelope contour.

\[
z = -B'D' \pm A'\sqrt{-\left(B'^2 + B\left(A'^2 + D'^2\right)\right)} - \frac{D}{C} \frac{(A'^2 + D'^2)}{C}
\]

(3.1.57)

\[
r = \sqrt{\frac{B' + (Cz + D)D'}{A'}} + A - z^2
\]

(3.1.56)

\[
A' = 2\left(d + h_2 \cos \alpha_2\right) \quad B' = -4a_1^2d \sin^2 \alpha_2
\]

\[
D' = -2a_1 \cos \alpha_2 \frac{\cos \alpha_1}{\sin \alpha_1}
\]
Fig. 2. Cross-section of a Cylindroid Ring workspace volume computed as an union of generating tori for a telescopic RRP manipulator with $a_1 = 3 \text{ u}$, $a_2 = 2 \text{ u}$, $d_1 = 0 \text{ u}$, $d_2 = 1 \text{ u}$, $\alpha_1 = 30 \text{ deg}$, $a_2 = 30 \text{ deg}$., $d_{\text{min}} = 1 \text{ u}$, $d_{\text{max}} = 6 \text{ u}$, ($\text{u}$ is a unit length).
Table 1 - Effect of twist angles $\alpha_1$ and $\alpha_2$ on the cross-section shape of workspace for telescopic RRP manipulators with $a_1 = 1\, \text{u}$, $a_2 = 2\, \text{u}$, $d_1 = 0\, \text{u}$, $d_2 = 1\, \text{u}$, $d_{\text{min}} = -5\, \text{u}$, $d_{\text{max}} = 5\, \text{u}$, ($\text{u}$ is a unit length). (see Fig.5).

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