Table 2 - Effect of the link parameters a_1 and a_2 on the cross-section shape of workspace for telescopic RRP manipulators with d_1 = 0 u, d_2 = 1 u, α_1 = 30 deg, α_2 = 30 deg., d_{min} = - 5 u, d_{max} = 5 u, (u is a unit length). (see Fig.6).

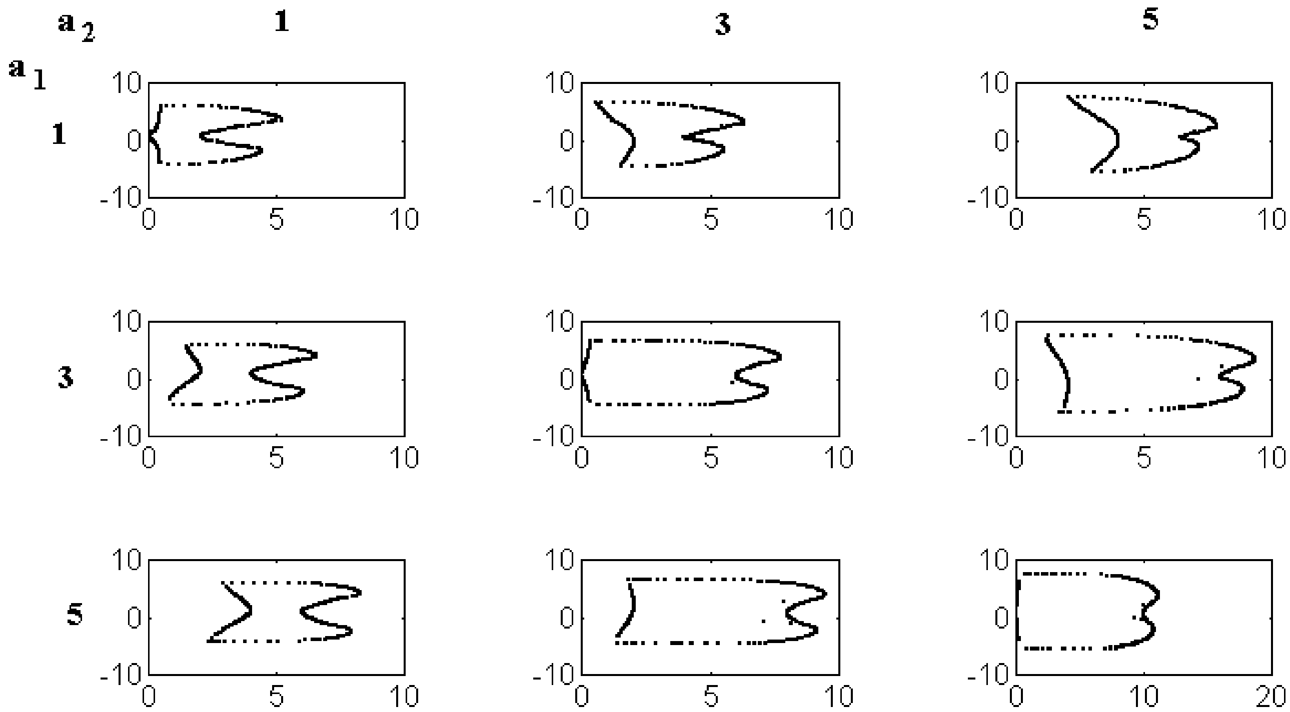
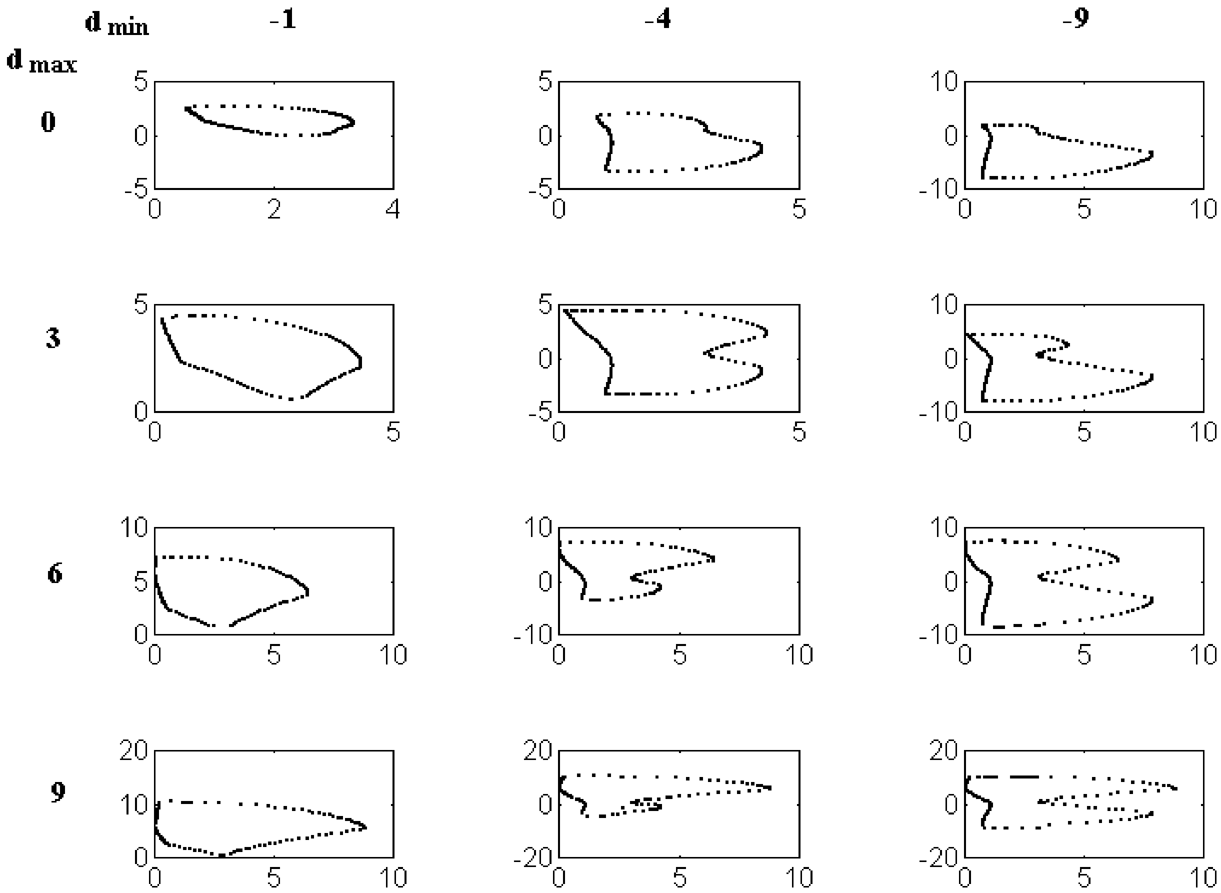


Table 3 - Effect of the stroke values d_{min} and d_{max} on the cross-section shape of workspace for telescopic RRP manipulators with a_1 = 1 u, a_2 = 2 u, d_1 = 0 u, d_2 = 1 u, α_1 = 30 deg, α_2 = 30 deg., (u is a unit length). (see Fig.7).



A workspace evaluation

by means of numerical simulations and/or experimental tests.

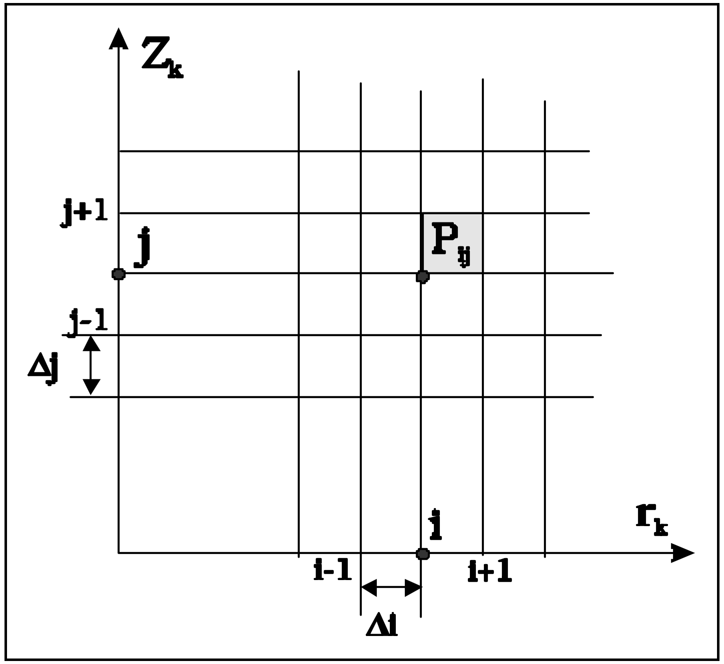
The fundamental characteristics of manipulator workspace for a numerical evaluation can be identified for both position and orientation capabilities as:

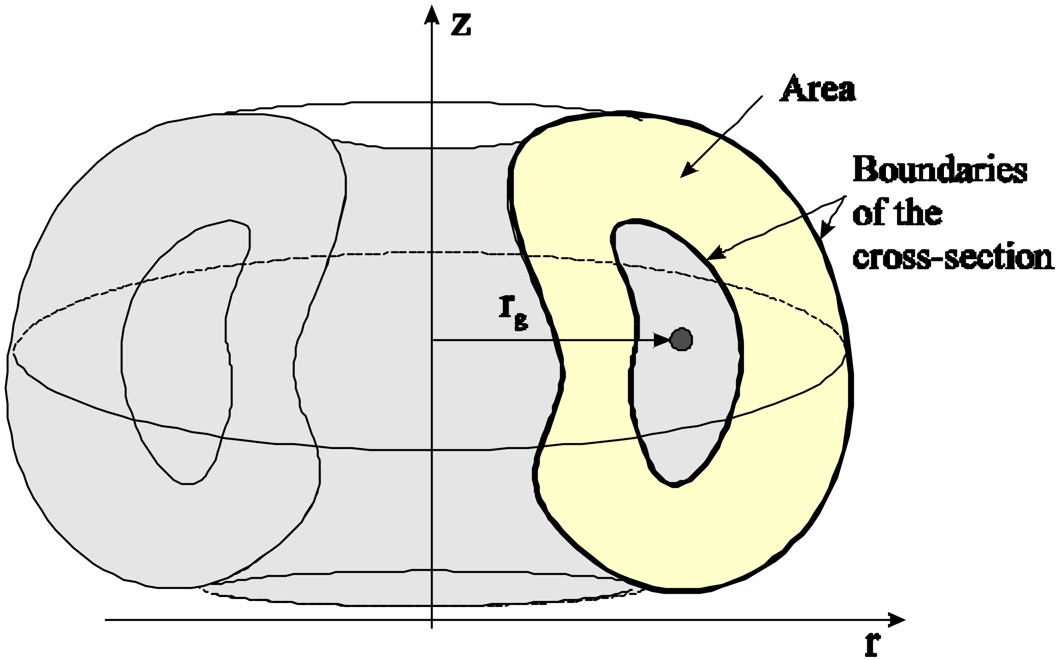
- shape and value of cross-section areas;
- shape and value of workspace volume;
- shape and extension of hole and voids;
- reach distances and reach ranges.

the repeatability measure through frequency matrix plots.

Once the workspace points (both in position and orientation) are determined,
one can use them to perform an evaluation of the above-mentioned workspace characteristics.

by using a grid evaluation or an algebraic formula.





$$A = \sum_{i=1}^{I} \sum_{j=1}^{J} \left({}_{A} P_{ij} \; \Delta i \, \Delta j \right) \tag{3.1.58}$$

$$V = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[P_{ijk} \; \Delta i \, \Delta j \, \Delta k \right] \tag{3.1.60}$$

By using the boundary contour points through the Pappus-Guldinus

$$V = 2\pi \sum_{i=1}^{I} \sum_{j=1}^{J} \left[P_{ij} \; \Delta i \, \Delta j \left(i \, \Delta i + \frac{\Delta i}{2} \right) \right] \tag{3.1.61}$$

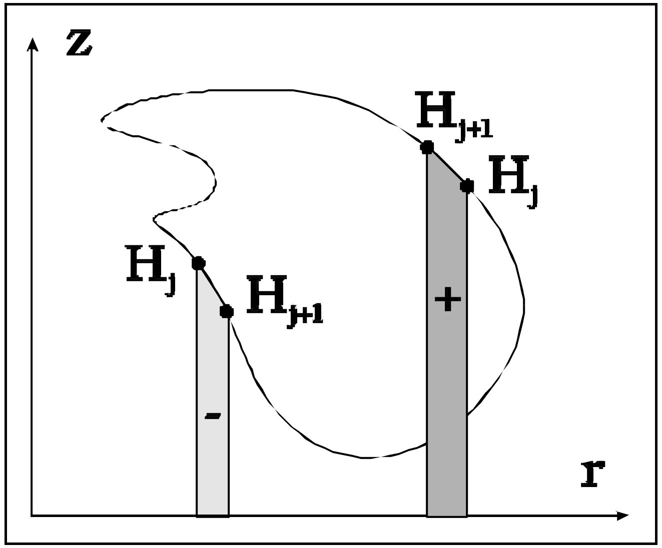


Fig.3.18: A scheme for the computation of workspace volume of serial open-chain manipulators with revolute joints.

$$A = \sum_{j=1}^{N} \left(z_{1,j+1} + z_{1,j}\right)\left(r_{1,j} - r_{1,j+1}\right) \qquad (3.1.59)$$

$$V = \frac{\pi}{2} \sum_{j=1}^{N} \left(z_{1,j+1} + z_{1,j}\right)\left(r_{1,j}^{2} - r_{1,j+1}^{2}\right) \qquad (3.1.62)$$

- Similarly, hole and void regions can be numerically evaluated by using Eqs. (3.1.58) to (3.1.62)
- Orientation workspace can be evaluated similarly by considering the angles in a Cartesian frame representation.

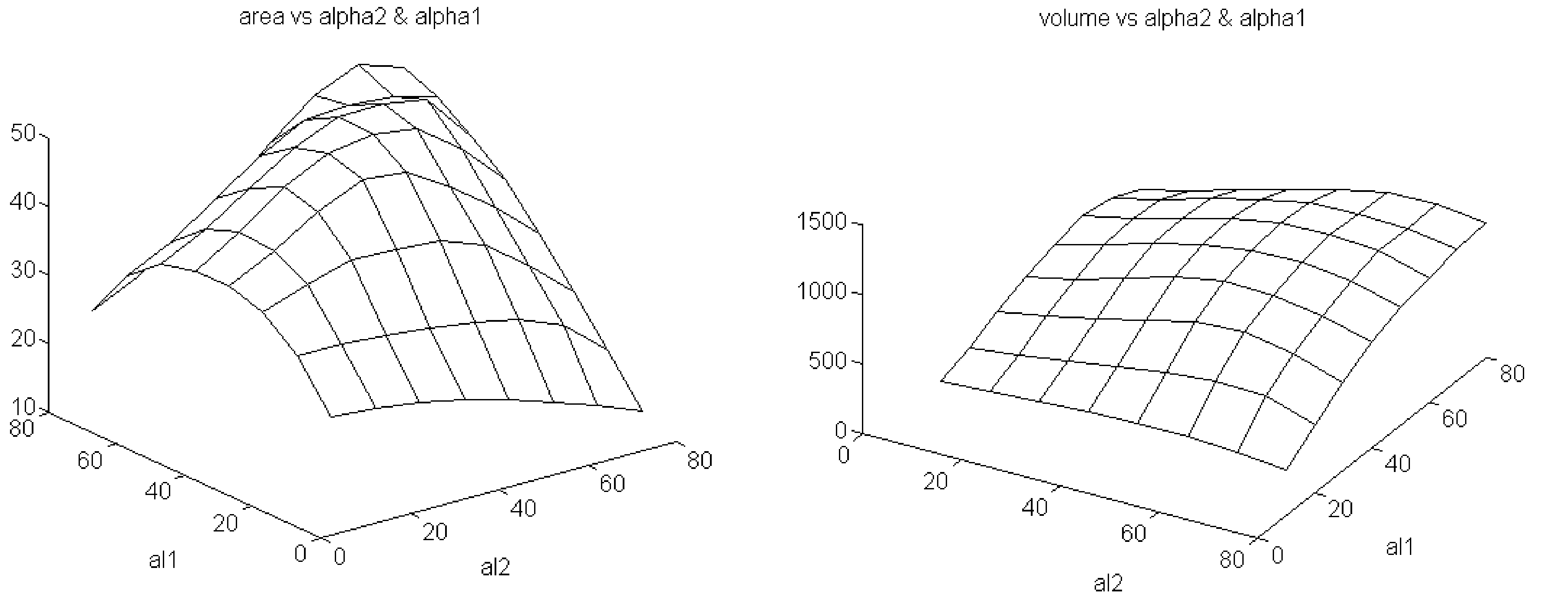


Fig.5. Effect of twist angles α_1 and α_2 on the cross-section area and the volume of workspace for telescopic RRP manipulators with a_1 = 1 u, a_2 = 2 u, d_1 = 0 u, d_2 = 1 u, d_{min} = - 5 u, d_{max} = 5 u, (u is a unit length). (see Table 1).

Manipulator design with prescribed workspace

- a general design problem can be formulated as finding the dimensions of a manipulator whose workspace cross-section is within or is delimited by the given axial and radial reaches r_{min}, r_{max}, z_{min}, z_{max}.

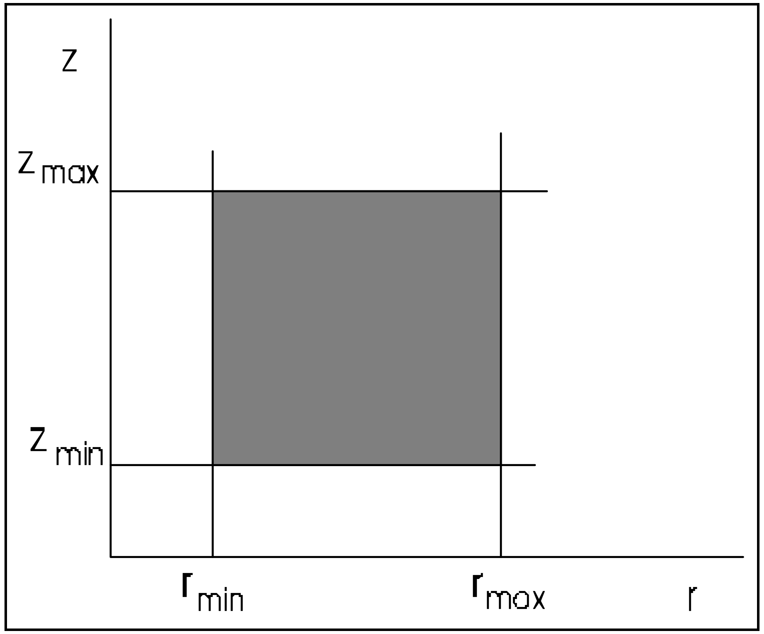


Fig.3.19: A general scheme for prescribing workspace limits of a manipulator.

In order to outline a design procedure, the case of a 3R manipulator is discussed in detail.

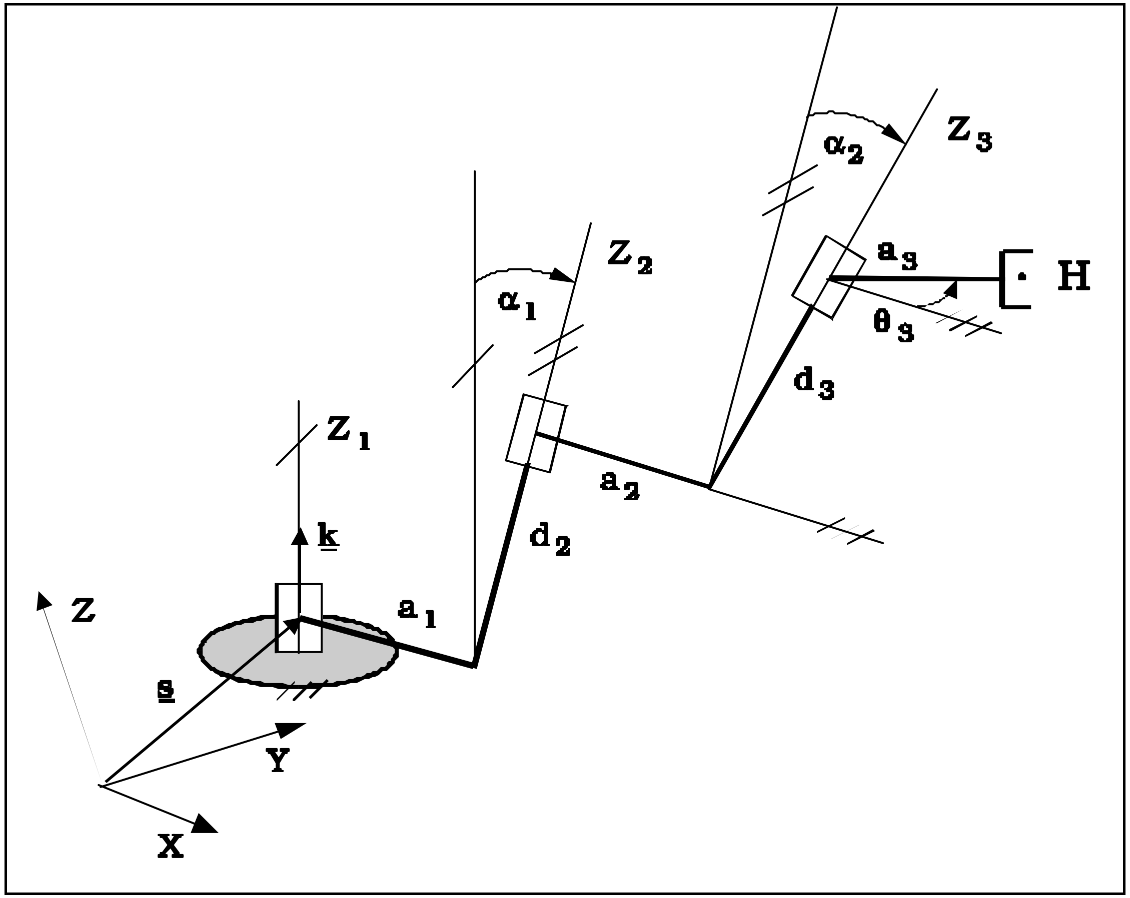


Fig.3.20: The kinematic chain of a general 3R manipulator and its design parameters.

A general open-chain 3R manipulator with three revolute joints is sketched in Fig.3.20, in which the design parameters are represented as the H-D **parameters a_1, a_2, a_3, d_2, d_3, a_1, a_2, and q_3** is the Z_3 joint variable.

$$r_1 = \left[A_1 - z_1^2 + \frac{(C_1 z_1 + D_1) G_1 + F_1}{E_1} \right]^{1/2} \qquad z_1 = \frac{-L_1 \pm Q_1^{1/2}}{K_1 C_1} - \frac{D_1}{C_1} \tag{3.1.63}$$

with **so-called structural coefficients** as

$$A_1 = a_1^2 + r_2^2 + (z_2 + d_2)^2 \qquad B_1 = -4a_1^2 r_2^2 \qquad C_1 = \frac{2a_1}{s\alpha_1} \qquad D_1 = -2a_1 (z_2 + d_2) \frac{c\alpha_1}{s\alpha_1}$$

with

$$r_2 = \left[(a_3 c\theta_3 + a_2)^2 + (a_3 s\theta_3 c\alpha_2 + d_3 s\alpha_2)^2 \right]^{1/2} \qquad z_2 = d_3\, c\alpha_2 - a_3\, s\theta_3\, s\alpha_2 \tag{3.1.65}$$

The remaining structural coefficients

$$E_1 = -2a_3 (d_2\, s\alpha_2 c\theta_3 + a_2\, s\theta_3) \qquad F_1 = 4a_1^2 a_3 (a_3\, s^2\alpha_2 s\theta_3 c\theta_3 + a_2 s\theta_3 - d_3 s\alpha_2 c\alpha_2 c\theta_3)$$

$$G_1 = 2a_1 a_3\, c\alpha_1 s\alpha_2 c\theta_3 / s\alpha_1 \qquad K_1 = G_1^2 + E_1^2 \qquad L_1 = F_1 G_1$$

$$Q_1 = L_1^2 - K_1 \left(F_1^2 + B_1 E_1^2 \right)$$

- Thus, the workspace boundary of a general three-revolute open-chain manipulator can be evaluated by scanning the angle θ_3 and plotting r_1, z_1.

Inversion of the formulation for workspace boundary

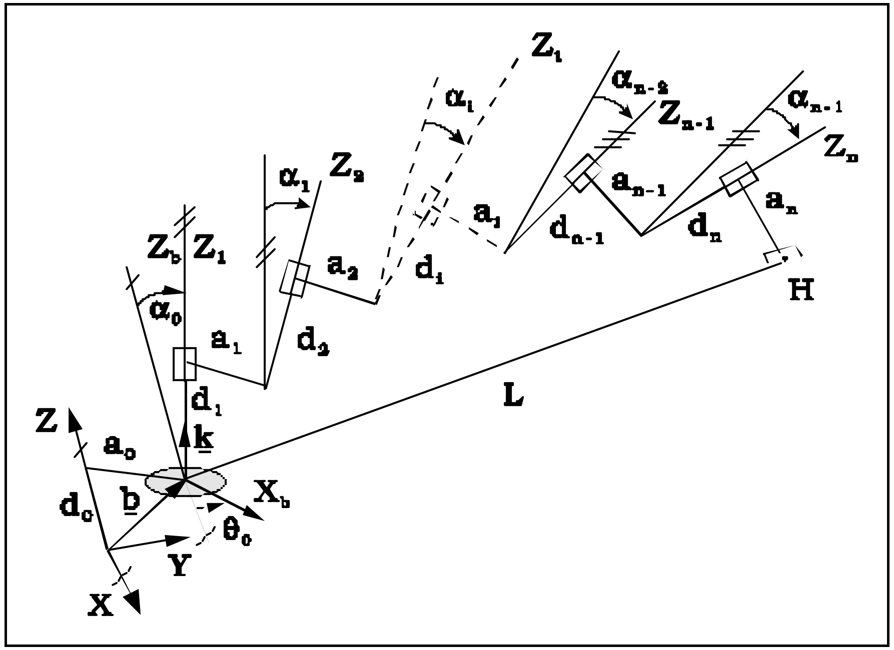


Fig.3.21: Design parameters for a general n-R manipulator.

Assuming as additional design parameters the position and orientation vectors **s** and **k** of the manipulator base with respect to the fixed world frame XYZ, Fig.3.20, the workspace design equations (3.1.63) can be modified to include the reference change. To accomplish this we take **x** as the position vector of a boundary point with respect to XYZ, and we use the expression

$$\mathbf{x}_1 = \mathrm{R}\mathbf{x} - \mathbf{s}_1 \qquad (3.1.67)$$

Introducing Eq.(3.1.67) into Eqs.(3.1.63), the result can be expressed, after some algebraic manipulations, in a vector form as

$$E_1(\mathbf{x}\cdot\mathbf{x}-2\mathbf{s}\cdot\mathbf{x}+\mathbf{s}\cdot\mathbf{s}-A_1)-\{C_1[\bar{r}_3\cdot(\mathbf{x}-\mathbf{s})]+D_1\}G_1+F_1=0 \qquad (3.1.68)$$

and

$$\mathbf{r}_3\cdot(\mathbf{x}-\mathbf{s})=\frac{-L_1\pm Q_1^{1/2}}{K_1C_1}-\frac{D_1}{C_1} \qquad (3.1.69)$$

where $\mathbf{r}_3$ is the third row vector of R; the Z_1 component of $\mathbf{s}_1$ can be computed as $s_{1z} = R\,\mathbf{k}_1\cdot R^t\,\mathbf{s}_1 = \mathbf{r}_3\cdot\mathbf{s}$; and $\mathbf{k}_1$ is the orientation vector of robot base as measured in $X_1Y_1Z_1$.

the design unknowns are represented by

- the link sizes,
- the structural coefficients A_1, B_1, C_1, D_1, E_1, F_1, G_1,
- and the manipulator base location vectors **s**, **k**.

- through the Newton-Raphson technique from a set of equations which express the workspace boundary points.

two decoupled set of design equations.

- Case: known robot base

to solve A_1, E_1, F_1, G_1, and B_1

$$E_1K_1\left(r_1^2 + z_1^2 - A_1\right) - \left(-L_1 \pm Q_1^{1/2}\right)G_1 - F_1K_1 = 0 \quad (3.1.70)$$

the remaining structural unknowns C_1 and D_1 can be evaluated by means of $z_1 = \dfrac{-L_1 \pm Q_1^{1/2}}{K_1C_1} - \dfrac{D_1}{C_1}$

- Case: unknown robot base

to solve the unknown A_1, E_1, F_1, G_1, Q_1 and **s**

$$E_1K_1\left(\mathbf{x}\cdot\mathbf{x} - 2\mathbf{s}\cdot\mathbf{x} + \mathbf{s}\cdot\mathbf{s} - A_1\right) - \left(-L_1 \pm Q_1^{1/2}\right)G_1 - F_1K_1 = 0 . \quad (3.1.71)$$

the remaining four boundary points to give C_1, D_1 and $\mathbf{r}_3$

$$\mathbf{r}_3 \cdot (\mathbf{x} - \mathbf{s}) = \frac{-L_1 \pm Q_1^{1/2}}{K_1C_1} - \frac{D_1}{C_1} \qquad r_{31}^2 + r_{32}^2 + r_{33}^2 = 1 \quad (3.1.72)$$

Since we are interested in **k**, $\mathbf{r}_3$ together with the orthonormal unit vector constraints can be used to evaluate the R matrix, whose third column represents **k** unit vector with respect to XYZ.

Once the structural coefficients are numerically determined, assuming $w = \sin^2\alpha_1$,

their expressions Eqs.(3.1.64) can be inverted to give

$$a_1 = 0.5\, C_1\, w^{1/2}$$

$$r_2^2 = -\frac{B_1}{w\, C_1^2} \qquad (3.1.73)$$

$$z_2 + d_2 = -\frac{D_1}{C_1 (1-w)^{1/2}}$$

and only the parameter w needs to be solved.

$$\alpha_1 = \pm \sin^{-1} w^{1/2} \qquad (3.1.81)$$

Substituting Eqs.(3.1.73) into the first of Eqs.(3.1.64), with the position $w = y + (1 + 4 A_1 / C_1^2) / 3$, it yields

$$y^3 + 3\,p\,y + 2\,q = 0 \qquad (3.1.74)$$

where

$$p = \frac{4}{9C_1^2}\left[A_1 - \frac{C_1^2}{4} - \frac{12}{C_1^2}\left(\frac{B_1 + D_1^2}{4} + \frac{A_1^2}{3}\right)\right] \qquad q = \frac{1}{27C_1^2}\left(1 + 4\frac{A_1}{C_1^2}\right)\left(10A_1 - 16\frac{A_1^2}{C_1^2} - 18\frac{B_1 + D_1^2}{C_1^2} - C_1^2\right) + 2\frac{B_1}{C_1^4}$$

(3.1.75)

Depending on the discriminant term $D = q^2 + p^3$ (3.1.76)

Equation (3.1.74) can be solved algebraically using Cardano's formula as function of the discriminant D:

- when $D > 0$, one real solution is expressed as $y = [\, q + D^{1/2}\,]^{1/3} + [\, q - D^{1/2}\,]^{1/3}$ (3.1.77)

- when $D = 0$ two real solutions are expressed as $y_1 = 2\, q^{1/3}$, $y_2 = -\, q^{1/3}$; (3.1.78)

- when $D < 0$ three real solutions are expressed as

$$y_1 = 2\, p^{1/2} \cos(u/3), \quad y_2 = 2\, p^{1/2} \cos(u/3 + 2\,\pi/3), \quad y_3 = 2\, p^{1/2} \cos(u/3 + 4\pi/3), \qquad (3.1.79)$$

where $u = \cos^{-1}(q/p^{3/2})$ (3.1.80)

Looking at the formulae (3.1.77) to (3.1.79) it is observable that Eq.(3.1.74) gives

one or two solutions of w according to the condition $0 < w < 1$

since at least one of the Eqs.(3.1.79) gives a negative value.

so that **each solution for w** corresponds to

- two manipulators distinguished at this step by the α_1 sign
- two more manipulators taking into account the supplementary values of α_1.

Successively, with the hypothesis that $\theta_3 = 0$, inverting Eqs.(3.1.65) and (3.1.66) the remaining chain parameters can be obtained as

$$d_2 = -\frac{E_1 a_1}{G_1 \tan\alpha_1} \qquad d_3 = \frac{z_2}{\cos\alpha_2} \qquad a_3 = \frac{G_1 \tan\alpha_1}{2a_1 \sin\alpha_2} \qquad a_2 = \left(r_2^2 - z_2^2 \tan^2\alpha_2\right)^{1/2} - a_3 \tag{3.1.82}$$

from the expression of the two-link length $d_3^2 + a_2^2 + a_3^2 = r_2^2 + z_2^2$ to give

$$\alpha_2 = \pm \sin^{-1}\left[\frac{U^2 + r_2^2 \pm \left[\left(U^2 - r_2^2\right)^2 - 4U^2 z_2^2\right]}{2\left(r_2^2 + z_2^2\right)}\right]^{1/2} \tag{3.1.83}$$

where $U = (G_1 \tan\alpha_1)/(2a_1)$.

each solution for α_2 corresponds to

- two manipulators distinguished
- two more manipulators taking into account the supplementary values .

each numerical solution for the structural coefficients and the manipulator base location corresponds

to sixty-four different manipulator parameter sets at the most,

depending on the number of solutions for α_1 and α_2.

However, **meaningful solutions can be considered only the sixteen sets,**

which can be synthesized for $-\pi/2 < \alpha_1 < \pi/2$ and $-\pi/2 < \alpha_2 < \pi/2$.

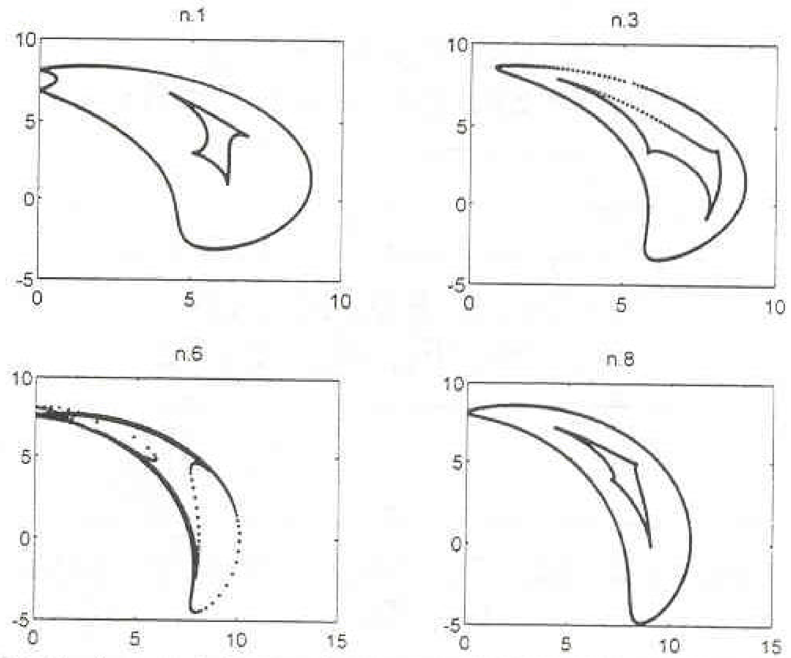


Fig. 3 A numerical example: some cross-sections of manipulator workspaces from Table 1

Table 1 A numerical example: manipulator parameters for workspaces with A = 69.12, B = -222.63, C = 4, D = -12.229, E = -13.646, F = -21.62, G = 12.00. The link dimensions are expressed in length unit and the angles in degree.

no.	a_1	α_1	d_2	α_2	d_3	a_3	a_2
1	1.00	30.00	1.97	35.96	1.93	5.90	1.47
2	1.00	30.00	1.97	-35.96	1.93	5.90	13.27
3	1.00	30.00	1.97	75.35	6.17	3.58	0.89
4	1.00	30.00	1.97	-75.35	6.17	3.58	8.06
5	1.00	-30.00	-1.97	30.51	6.38	6.82	13.54
6	1.00	-30.00	-1.97	-30.51	6.38	6.82	-0.10
7	1.00	-30.00	-1.97	51.25	8.79	4.44	7.38
8	1.00	-30.00	-1.97	-51.25	8.79	4.44	1.49
9	1.80	64.00	1.00	12.27	6.12	32.20	28.25
10	1.80	64.00	1.00	-12.27	6.12	32.20	36.14
11	1.80	64.00	1.00	--	--	--	--
12	1.80	64.00	1.00	--	--	--	--
13	1.80	-64.00	-1.00	27.30	8.97	14.92	15.46
14	1.80	-64.00	-1.00	-27.30	8.97	14.92	14.38
15	1.80	-64.00	-1.00	--	--	--	--
16	1.80	-64.00	-1.00	--	--	--	--

Optimum design formulation

An optimum design of manipulators can be formulated as an optimization problem in the form

$$\min f \qquad (3.1.84)$$

subject to

$$\mathbf{x}_j \geq \mathbf{X}_j \qquad (j = 1,\ldots,J) \qquad (3.1.85)$$

$$V \geq V_0 \qquad (3.1.86)$$

where

- f is the objective function;
- $\mathbf{X}_j$ (j=1,...,J) represent given precision workspace points,
- V_0 is a minimum value for a desirable workspace volume.

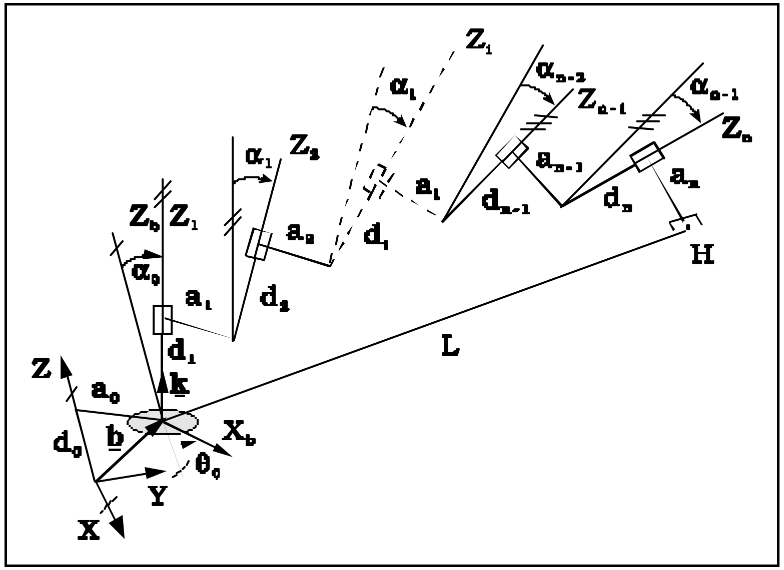


Fig.3.21: Design parameters for a general n-R manipulator.

objective function

Several workspace characteristics can be used to formulate the objective function **but however workspace volume and manipulator length are usually preferred.**

the volume V of a manipulator workspace is related to the manipulator length L in a sense that larger is the manipulator when larger is workspace volume is obtainable.

$$V = c\, L^{\beta} \tag{3.1.87}$$

where β is a constant and c is a function of the chain parameters;
the manipulator length L can be defined as

$$L = \sqrt{\sum_1^n (a_i^2 + d_i^2)} \tag{3.1.88}$$

that depends of the link ratios and dimensions, respectively in the form

$$L = a_1 \sqrt{\sum_1^{2n+1} k_i^2} \tag{3.1.89}$$

where k_i , i=1,...,2n+1, are the link ratios of a_i and d_i , i=1,...,n, with respect to a_1.

Consequently, the workspace volume can be computed by introducing Eq.(3.1.88) in the expression for a volume of revolution to give

$$V = \frac{2\pi}{\sqrt[3]{\sum_{1}^{2n+1} k_i^2}} L^3 \oint_{\partial W} r^{*2}\, dz^* \qquad (3.1.90)$$

the so-called manipulator global length L_{tot}, which can be defined in the form

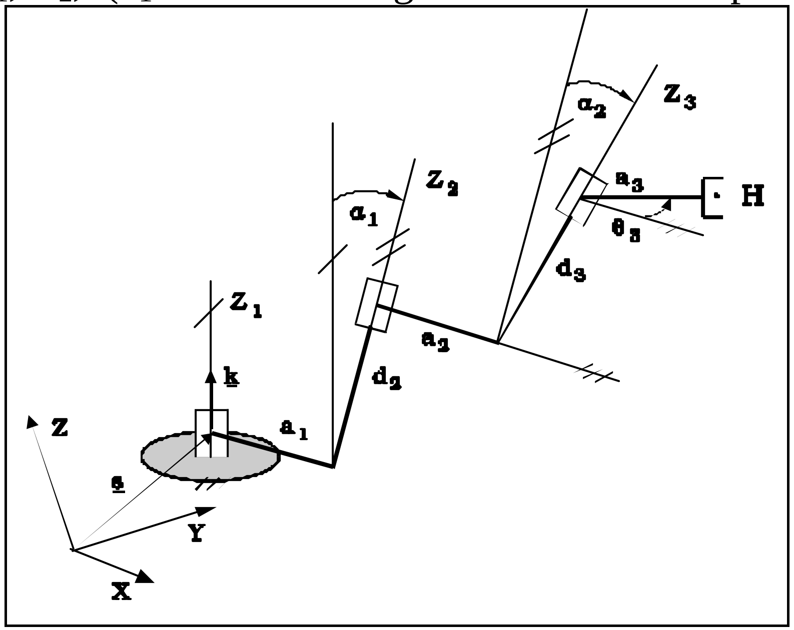
$$L_{tot} = \sum_{0}^{n} (a_i + d_i) \qquad (3.1.91)$$

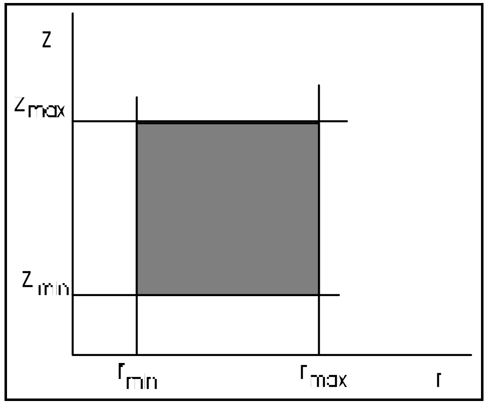
These characteristics can be also conveniently combined to formulate a performance index for manipulators.

like for example $$f = \frac{V^*}{L^3}$$

The general optimization formulation can be better illustrated by referring to a specific case

Thus, the optimum design of **a general three-revolute manipulator** deals with the synthesis of the parameters a_1, a_2, a_3, d_2, d_3, α_1, α_2, (d_1 is not meaningful since it shifts up and down the workspace only),.





$$\min -\frac{V^*}{L^3} \qquad (3.1.92)$$

subject to

$$\min(z) \geq z_{min} \qquad \max(z) \leq z_{max} \qquad \min(r) \geq r_{min} \qquad \max(r) \leq r_{max} \quad (3.1.93)$$