A novel dynamic programming algorithm for track-before-detect in radar systems

Emanuele Grossi, Member, IEEE, Marco Lops, Senior Member, IEEE and Luca Venturino, Member, IEEE

Abstract—In this paper we present a novel procedure for multi-frame detection in radar systems. The proposed architecture consists of a pre-processing stage, which extracts a set of candidate alarms (or plots) from the raw data measurements (e.g., this can be the Detection and Plot-Extraction stage of common radar systems), and a track-before-detect (TBD) processor, which jointly elaborates observations from multiple scans (or frames) and confirms reliable plots. A computationally efficient dynamic programming algorithm for the TBD processor is derived, which does not require a discretization of the state space and operates directly on the input plot-lists. Finally, a simple algorithm to solve possible data association problems arising at the track-formation step is given, and a thorough complexity and performance analysis is provided, showing that large detection gains with respect to the standard radar processing are achievable with negligible complexity increase.

Index Terms—Multi-frame detection (MFD), track-before-detect (TBD), dynamic programming, radar systems, Detector, Plot Extractor, tracking.

I. INTRODUCTION

The classical approach to radar detection and tracking follows the scheme reported in Fig. 1. At each scan (or frame), the raw data from the sensor undergo a processing chain in the Detection and Plot-Extraction block: the output measurements, called plots, are the result of a clustering operation to merge signal returns which appear to come from the same object, a constant false alarm rate filtering to mitigate clutter, and a thresholding operation retaining only “significant” (i.e., of sufficient strength) plots. If necessary, target tracking procedures, such as Kalman filtering or multiple hypothesis tracking, may be implemented in order to estimate targets’ trajectories on the basis of surviving plots [1].

The traditional approach shows some limitations in remote radar surveillance, where the signal amplitude is weak compared to the background noise, and, in general, in the detection of dim, fluctuating targets. Multi-frame detection (MFD) is a method to improve the detection of weak targets by integrating their signal returns over multiple consecutive scans. MFD is challenging in the presence of target motion, as track-before-detect (TBD) techniques are required to correctly integrate the echoes along the unknown target trajectory. Two main approaches to the MFD problem have been considered in the literature. In an atomistic approach [2]–[4], the integration simply takes place in the sensor measurement space: here the goal is to obtain more reliable detections to be sent to the subsequent, distinct tracking stage, which combines the positions (or the estimated track segments) into long continuous tracks. In a holistic approach [5]–[7], instead, the detection and tracking stages are fully merged: energy integration takes place in the target state space, and the estimated trajectory is returned at the same time as detection is declared.1 Starting from early studies on this topic [5], [9]–[12], a number of applications and improvements have been proposed with reference to both passive [6]–[8], [13]–[17] and active sensors [18]–[29], possibly accounting for some prior information on the target motion, and/or considering an unknown number of targets, and/or adopting sequential data processing [30]–[39].

State-of-the-art TBD strategies, however, hardly lead to real-time implementable schemes in the presence of highmobility targets when the cardinality of the state space is large, even resorting to dynamic programming algorithms, such as the Viterbi algorithm [40]. The main reason for such an un-affordable complexity is that all of the observations are retained at each epoch and processed. Another drawback of procedures elaborating raw data measurements is that the presence of many weak returns could result in a lower measurements’ accuracy. To overcome complexity limitations, the authors of [20], [25], [41] envisioned the adoption of a pre-processing stage. In [20], only the measurements whose signal amplitude exceeds a primary threshold are elaborated in the subsequent stage based on the Hough transform. In [41], the input data undergoes a one-bit quantization, and the multi-scan processing is simply a 3-D matched filter. In [25], instead, a multiple-bit quantization is considered, and the subsequent stage implements the Viterbi algorithm. None of these studies, however, take into account the fact that, at each epoch, the set of candidate detections may shrink or expand based on the significance of the received returns.

1As pointed out in [8]. TBD should be more properly defined as track-before-declare in this case, since target is tracked before declaring it to be a valid target. The term “detect,” instead, may generate confusion, in that it could be referred to target detections (i.e., measurements, alarms).
In this paper, we follow an atomistic approach to MFD
and, starting from preliminary results obtained in [42]–[45],
we propose a two-stage detection architecture. The first stage
is the classical Detector and Plot-Extractor, but the threshold
is lowered in order to obtain a richer set of candidate plots.
The second stage, instead, is a TBD processor which exploits
the space-time correlation among the candidate plots taken
at different scans to confirm or delete them.\(^2\) The main
contribution is the derivation of a novel dynamic programming
algorithm to compute the test statistics needed in the second
stage. The algorithm operates directly on the candidate plots
and efficiently takes into account the fact that their number is,
in general, much smaller than that of resolution elements. A
complexity analysis is provided, stating the conditions under
which the proposed procedure outperforms its direct competi-
tor, i.e., the Viterbi algorithm. Finally, a simple algorithm to
solve possible data association problems arising at the track-
formation step is given, and a thorough performance assess-
ment is undertaken to elicit the trade-offs among the primary
threshold, the achievable performance, and the computational
complexity.

The reminder of the paper is organized as follows. In the
next section, the two-stage detection architecture is described.
In Sec. III the novel TBD algorithm is presented, and its per-
formance is analyzed in Sec. IV. Finally, concluding remarks
are given in Sec. V.

II. TWO-STAGE MULTI-FRAME DETECTION

The proposed two-stage approach follows the scheme out-
lined in Fig. 2. At each scan \(n\), the Detection and Plot-
Extraction stage receives the raw data collected by the sen-
or and produces a list of candidate plots (or alarms). The
processing typically concerns a detection, a clustering, and an
extraction step, and operates with a primary threshold \(\gamma_1\). The
\(k\)-th plot at scan \(n\) is the 5-dimensional vector
\[
s_{k,n} = \left( t_{k,n}, r_{k,n}, \theta_{k,n}, A_{k,n}, N_{k,n} \right)
\]
where \(t_{k,n}\) is the time instant at which the alarm has been
taken, \(r_{k,n}\) is the range measurement, \(\theta_{k,n}\) is the azimuth
measurement, \(A_{k,n}\) is the amplitude of the received signal,
and \(N_{k,n}\) is the power of the disturbance (thermal noise plus
clutter). These are the typical measurements taken by a long-
range surveillance radar system, but additional measurements,
such as elevation angle and/or range-rate, can be taken into
account. We also assume that the radar is ground-based, but
the discussion can be easily generalized to ship- and air-borne
radars by compensating the (known) platform motion.

The plots at scan \(n\), whose number is denoted \(D_n\), are
organized in a plot-list, which is the \(D_n \times 5\) matrix defined,
for \(D_n \neq 0\), as
\[
S_n = \begin{pmatrix}
\mathbf{s}_{1,n} \\
\vdots \\
\mathbf{s}_{D_n,n}
\end{pmatrix}
\]
The plot-lists corresponding to the current and the previous
\(L - 1\) scans are the input of the second stage. After examining
the correlation among the alarms in \(\{S_{\ell}\}_{\ell=n-L+1}^{n}\), the second
stage confirms or deletes each alarm contained in the current
plot-list \(S_n\) through a secondary threshold \(\gamma_2\).

This operating scheme is extremely flexible and versatile,
and it subsumes both the traditional scheme in Fig. 1, if the
memory of the second stage is set equal to \(L = 1\), and MFD
with raw input data [2]–[4], [25], if the primary threshold
\(\gamma_1 = -\infty\). In its typical operating mode, the first stage adopts
a primary threshold lower than that used in the traditional
scheme in Fig. 1, and this causes an increment in both the
probability of detection (PD), which is the probability that
an alarm is true, and the false alarm rate (FAR), which is
the average number of false alarms in a minute. The goal of
the second stage is to restore the FAR to the level originally
granted by the traditional scheme through a higher threshold
value while maintaining part (if not all) of the gain in terms of
PD, or, alternatively, to obtain a gain in terms of FAR
maintaining the same PD as the traditional scheme.

In the following, the probability that an alarm is false will
be referred to as PFA, and the subscripts “in/out” will be added
to PD, PFA, and FAR to specify whether these quantities refer
to the plot-list at the input or at the output of the second stage,
respectively. In the reminder of the section we present the test
statistics employed in the second stage and the target kinematic
constraints involved, while in Sec. III we illustrate in detail the
whole processing chain carried out by the second stage.

A. Test statistics

To simplify exposition, let us assume that \(S_L\) is the current
plot-list, so that the observations taken at scans \(\ell = 1, \ldots, L\)
are jointly processed. The task of the second stage is to
confirm or delete \(s_{k,L}\), for \(k = 1, \ldots, D_L\). The trajectory
of a prospective target from scan 1 to L can be specified by
an \(L\)-dimensional vector, say \(\mathbf{\nu} = (\nu_1, \ldots, \nu_L)\), with
\(\nu_\ell \in \{0, 1, \ldots, D_\ell\}\) for \(\ell = 1, \ldots, L\). Specifically, \(\nu_\ell = k\)
means that the target is observed at scan \(\ell\), and the corre-
sponding alarm is \(s_{k,\ell}\), while \(\nu_\ell = 0\) that there is a missing
observation at scan \(\ell\). The sequence of plots indexed by \(\mathbf{\nu}\)
is \(\{s_{\nu_\ell,\ell} : \ell = 1, \ldots, L \text{ and } \nu_\ell \neq 0\}\).

Define, for \(\ell = 1, \ldots, L\),
\[
z_{k,\ell} = \begin{cases}
A_{k,\ell}^2/N_{k,\ell}, & \text{if } k \in \{1, \ldots, D_\ell\} \\
\eta, & \text{if } k = 0
\end{cases}
\]
range and azimuth errors, respectively, and assume that these errors are zero-mean; then the standard deviations of the errors on the velocities in (4) are approximately equal to

$$\sigma_{v_r,2} = \sqrt{\frac{(1 + \cos^2(\theta_2 - \theta_1))\sigma_r^2 + 2r_2^2 \sin^2(\theta_2 - \theta_1)\sigma_\theta^2}{t_2 - t_1}}$$

$$\sigma_{v_t,2} = \sqrt{\frac{\sin^2(\theta_2 - \theta_1)\sigma_r^2 + 2r_2^2 \cos^2(\theta_2 - \theta_1)\sigma_\theta^2}{t_2 - t_1}}$$

and the velocity constraint becomes

$$\left[\left|v_{r,2} - \beta \sigma_{v_r,2}\right|ight]^2 + \left[\left|v_{t,2} - \beta \sigma_{v_t,2}\right|ight]^2 < v_{\text{max}}^2$$

where $$x^+ = \max\{x, 0\}$$, and $$\beta$$ accounts for a given percentage of the errors.\(^4\)

If measurements $$(t_2, r_2, \theta_2)$$ and $$(t_1, r_1, \theta_1)$$ pass the velocity check, and if there is a previous measurement $$(t_0, r_0, \theta_0)$$ in the trajectory, then the target acceleration can be also checked by comparing the mean velocities in (4) with the mean velocities for $$(t_1, r_1, \theta_1)$$, say $$v_{r,1}$$ and $$v_{t,1}$$, obtained from the measurements $$(t_1, r_1, \theta_1)$$ and $$(t_0, r_0, \theta_0)$$ (see Fig. 3). Hence, the radial and tangential accelerations for $$(t_2, r_2, \theta_2)$$ are

$$a_{r,2} = \frac{v_{r,2} - \left(v_{r,1} \cos(\theta_2 - \theta_1) + v_{t,1} \sin(\theta_2 - \theta_1)\right)}{t_2 - t_1} - \frac{r_2 - r_1 \cos(\theta_2 - \theta_1)}{(t_2 - t_1)^2}$$

(5a)

$$a_{t,2} = \frac{v_{t,2} - \left(-v_{r,1} \sin(\theta_2 - \theta_1) + v_{t,1} \cos(\theta_2 - \theta_1)\right)}{t_2 - t_1} - \frac{r_1 \sin(\theta_2 - \theta_1) - r_0 \cos(\theta_2 - \theta_0)}{(t_2 - t_1)(t_1 - t_0)}$$

(5b)

Taking into account the errors on the radial and tangential acceleration, whose approximated expressions of the standard deviations are given in (6), the acceleration constraint becomes

$$\left[\left|a_{r,2} - \beta \sigma_{a_r,2}\right|ight]^2 + \left[\left|a_{t,2} - \beta \sigma_{a_t,2}\right|ight]^2 < a_{\text{max}}^2$$

We finally consider an additional constraint on the maximum number of consecutive misses in the candidate trajectories, say $$P \in \{0, \ldots, L-1\}$$. This ensures that trajectories with large “holes,” are not examined: this is desirable since consecutive plots too spaced away in time are scarcely correlated. Clearly, consecutive misses at the beginning of the trajectory should be allowed, since they account for newly born targets, i.e., targets that enter the scene during the $$L$$ elaborated scans.

\(^3\)Recall that the standard deviation of a non-linear function $$f$$ of the uncorrelated random variables $$x_1, \ldots, x_n$$ with standard deviations $$\sigma_{x_1}, \ldots, \sigma_{x_2}$$ is approximately equal to $$\sigma_f = (\sum_{i=1}^{n} (\partial f/\partial x_i)^2 \sigma_{x_i}^2)^{1/2}$$ [46].

\(^4\)E.g., in the Gaussian case, about 95.5% and 99.7% of the error values are within $$\beta = 2$$ and $$\beta = 3$$ standard deviations from the mean, respectively. If the distribution of the errors is not known, then Chebyshev’s inequality can be used: e.g., the amount of data within $$\beta = 2$$ or $$\beta = 3$$ standard deviations from the mean is always at least 75% or 89%, respectively.
\[
\sigma_{a_{r,2}} = \frac{1}{t_2 - t_1} \left\{ \frac{1}{(t_2 - t_1)^2} + \cos^2(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{\cos^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \right\} \sigma_r^2
\]
\[+ \left( r_1 \sin(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right) - \frac{r_0 \sin(\theta_2 - \theta_0)}{t_1 - t_0} \right)^2 \sigma_r \]
\[+ r_1^2 \sin^2(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{r_0^2 \sin^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \left( \sigma_r \right)^2 \right\}^{1/2}
\]
\[
\sigma_{a_{t,2}} = \frac{1}{t_2 - t_1} \left\{ \sin^2(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{\sin^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \right\} \sigma_r^2
\]
\[+ \left( r_1 \cos(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right) - \frac{r_0 \cos(\theta_2 - \theta_0)}{t_1 - t_0} \right)^2 \sigma_r \]
\[+ r_1^2 \cos^2(\theta_2 - \theta_1) \left( \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \right)^2 + \frac{r_0^2 \cos^2(\theta_2 - \theta_0)}{(t_1 - t_0)^2} \left( \sigma_r \right)^2 \right\}^{1/2}
\]

Therefore, denoting \(z(\nu)\) the maximum number of consecutive zeros after the first non-zero entry of the vector \(\nu\), we require that all \(\nu \in R_{k,\ell}\) satisfy \(z(\nu) \leq P\).

### III. ALGORITHM DESCRIPTION

The proposed TBD processor in Fig. 2 consists of four distinct blocks, as shown in Fig. 4:

i) **Track Formation**, which computes the test statistics in (3) accounting for the measurement accuracy and the constraints of Sec. II-B;

ii) **Track Pruning**, which solves the data-association problem arising when multiple estimated trajectories share a common root;

iii) **Plot Confirmation**, which compares the decision statistics with the threshold \(\gamma_2\) and confirm or delete the plots accordingly;

iv) **Track Smoothing** (optional), which improves the measurement accuracy of the confirmed plots.

In the following each block is described in detail.

#### A. Track Formation

In most radar applications, the cardinality of the set of physically-admissible candidate trajectories is very large, and a brute-force, exhaustive search is not feasible, as it would be exponentially complex in the number of integrated scans. A possible way to reduce complexity from exponential to linear in the number of integrated scans is to discretize the covered area and resort to the Viterbi algorithm [4]–[7], [25]. However, this approach can be still too demanding, since the number of resolution elements can be very large (of the order of \(10^5\) or more in many applications). The track formation algorithm presented next avoids discretization of the covered area and allows to compute the statistics in (3) with a complexity that is affordable in most applications.

Let \(R_{k,\ell}\) be the set of \(\ell\)-dimensional vectors indexing the admissible (i.e., satisfying the constraints of Sec. II-B) trajectories ending in \(s_{k,\ell}\) at scan \(\ell\), and define

\[
\tau_{k,\ell} = \arg \max_{\nu \in R_{k,\ell}} \sum_{p=1}^{\ell} z_{\nu_p, p}
\]

\[
F_{k,\ell} = \max_{\nu \in R_{k,\ell}} \sum_{p=1}^{\ell} z_{\nu_p, p}
\]

for \(k = 1, \ldots, D_r\), and \(\ell = 1, \ldots, L\), so that the test statistics in (3) are just \(\{F_{k,\ell}\}_{k=1}^{D_r}\). Also, let \(\mathcal{M}_{k,\ell}\) denote the set of indexes addressing all past alarms compatible (i.e., satisfying the constraints of Sec. II-B) with alarm \(k\) at scan \(\ell\), i.e.,

\[
\mathcal{M}_{k,\ell} = \{ (j, p) : p \in \{ \max\{1, \ell - P\}, \ldots, \ell - 1\}, j \in \{1, \ldots, D_p\}, \text{ and } s_{k,\ell} \text{ and the plots indexed by } \tau_{j,p} \text{ satisfy the kinematic constraints} \}
\]
Algorithm 1 Computes \( \{\tau_{k,\ell}, F_{k,\ell}\}_{k=1}^{D_L} \)

1. for \( k = 1, \ldots, D_1 \) do
2. \( F_{k,1} = z_{k,1} \)
3. \( \tau_{k,1} = k \)
4. end for
5. for \( \ell = 2, \ldots, L \) do
6. for \( k = 1, \ldots, D_\ell \) do
7. if \( \mathcal{M}_{k,\ell} \neq \emptyset \) then
8. \( (h, m) = \arg \max_{(j,p) \in \mathcal{M}_{k,\ell}} F_{j,p} \)
9. \( F_{k,\ell} = F_{h,m} + (l - m)\eta + z_{k,\ell} \)
10. \( \tau_{k,\ell} = (\tau_{h,m} 0 \cdots 0 k) \)
11. else
12. \( F_{k,\ell} = (\ell - 1)\eta + z_{k,\ell} \)
13. \( \tau_{k,\ell} = (0 \cdots 0 k) \)
14. end if
15. end for
16. end for

Once the algorithm is terminated, plot \( s_{k,L} \) along with the associated pruned trajectory \( \tau_{k,L} \) and test statistic \( F_{k,L} \) is sent to the Plot Confirmation stage.

Algorithm 2 Prunes \( \{\tau_{k,L}, F_{k,L}\}_{k=1}^{D_L} \) and recomputes \( \{F_{k,L}\}_{k=1}^{D_L} \)

1. \( W = \{1, \ldots, D_L\} \)
2. while \( W \neq \emptyset \) do
3. \( q = \arg \max_{w \in W} F_{w,L} \)
4. for \( p \in W: p \neq q \) do
5. \( \ell = 1 \)
6. while \( (\tau_{p,\ell}) = (\tau_{q,\ell}) \) do
7. \( \ell = \ell + 1 \)
8. end while
9. if \# of non-zero entries of \( \tau_{p,L} < Q \) then
10. \( \tau_{p,L} = (0 \cdots 0 p) \)
11. end if
12. \( F_{p,L} = \sum_{\ell=1}^{L} z(\tau_{p,L})_\ell \)
13. end for
14. \( W = W \setminus \{q\} \)
15. end while

B. Track Pruning

A data association problem may arise after computing the statistics in (3). Indeed, several estimated trajectories may share a common root, and true target echoes may be responsible not only for the confirmation of the true alarms they caused, but also of false alarms in their proximity. This ambiguity is solved by the Track Pruning stage, which executes Algorithm 2, where \( (v)_\ell \) denotes the \( \ell \)-th entry of the vector \( v \). The algorithm assigns the common root only to the trajectory with the largest test statistic (line 3) and all other trajectories are pruned accordingly (lines 5–9). Moreover, if the number of non-zero entries of a trajectory is smaller than a specified minimum value, say \( Q \), then the trajectory is considered unreliable, and only the final plot is maintained (line 11). Finally, all test statistics corresponding to the new shortened trajectories are recomputed (line 13).

This algorithm is an extension of the procedure introduced in [35, which has been preferred among the many proposed in the past as it offers a good compromise between complexity and performance.

Once the algorithm is terminated, plot \( s_{k,L} \) along with the associated pruned trajectory \( \tau_{k,L} \) and test statistic \( F_{k,L} \) is sent to the Plot Confirmation stage.

C. Plot Confirmation

Each plot in the current plot-list \( S_L \) is confirmed or deleted by comparing the corresponding decision statistic with the secondary threshold \( \gamma_2 \), i.e.,

\[
F_{k,L} \begin{cases} \geq \gamma_2 & \Rightarrow \text{confirm } s_{k,L} \\ < \gamma_2 & \Rightarrow \text{delete } s_{k,L} \end{cases}
\]

for all \( k \in \{1, \ldots, D_L\} \). Confirmed plots and their associated trajectories are sent, if needed, to the Track Smoothing stage.

D. Track Smoothing

Standard linear regression (or quadratic, if large maneuvers are expected) can be applied to confirmed plots bearing an estimated trajectory to improve the accuracy of range and azimuth measurements. Notice that, as a side result, the regression can also give information about velocity.

E. Complexity analysis

The computational complexity of the scheme in Fig. 4 is ruled by the complexity of Algorithm 1, which is a function of the number of integrated scans and of the number of plots per scan. The innermost loop of the algorithm requires to evaluate the set \( \mathcal{M}_{k,\ell} \), which amounts to check the kinematic constraints between \( s_{h,m} \) and \( s_{j,p} \), for all \( p = \max\{1, \ell - P\}, \ldots, \ell - 1 \), and \( j = 1, \ldots, D_p \). Therefore, the number of operations required in Algorithm 1 is in the order of

\[
\sum_{\ell=2}^{L} \sum_{p=\max\{1, \ell - P\}}^{L-1} D_p.
\]

Notice now that \( \{D_1, \ldots, D_L\} \) can be assumed to be a sequence of independent and identically distributed random variables. Specifically, denoting \( N_r \) and \( N_a \) the number of
resolution elements in range and azimuth, respectively, and $K \in \{0, 1, \ldots, N_r N_a\}$ the number of targets present in the scene, then each $D_\ell$ can be modeled as the sum of two independent Binomial random variables with parameters $(N_r N_a - K, PFA_{in})$ and $(K, PD_{in})$. Thus, the average number of required operations is on the order of

$$
\mathbb{E} \left[ \sum_{\ell=1}^L D_\ell \sum_{p=\max\{1, \ell-P\}}^{\ell-1} D_p \right]
$$

$$
= \sum_{\ell=1}^L \sum_{p=\max\{1, \ell-P\}}^{\ell-1} \left[ (N_r N_a - K)PFA_{in} + KPD_{in} \right]^2
$$

$$
= P \left( L - \frac{P + 1}{2} \right) \left[ (N_r N_a - K)PFA_{in} + KPD_{in} \right]^2
$$

where $\mathbb{E}$ denotes statistical expectation. Hence, the average complexity of Algorithm 1 when $K$ targets are present in the scene is $\mathcal{O}(LP[(N_r N_a - K)PFA_{in} + KPD_{in}]^2)$.

The formal definition of $PD_{out}$ of the second stage, and the subscripts “in” and “out” will be used to avoid confusion. The performance measures used to test the proposed scheme in Fig. 2 are PD and FAR at both the input and the output of the second stage, and the subscripts “in” and “out” will be used to avoid confusion. The formal definition of $PD_{out}$ is slightly different from that of $PD_{in}$. The latter has a “local” meaning and is defined as the probability that an alarm in the input plot-list is true. The former, instead, has a “global” meaning: specifically, since every alarm in the output plot-list bears a trajectory, $PD_{out}$ is the probability that the trajectory corresponding to an alarm in the output plot-list is true, i.e., caused by a target actually present in the scene. The root mean square error (RMSE) on the estimation of the target position is also considered, and, again, a subscript is added to address the input and the output. It is defined as

$$
RMSE = \sqrt{\mathbb{E} \left[ e^2(s_{k,L}) \mid H_1 \right]}
$$

where $H_1$ is the event that $s_{k,L}$ is confirmed by the second stage, and that its trajectory is true, and $e(s_{k,L})$ is the Euclidean distance between the true and the estimated target position.

We discuss a numerical example, where a Swerling I fluctuation model is assumed. The amplitude of the echoes are generated according to a Rayleigh distribution, so that the variables $\{z_{k,\ell} : \ell = 1, \ldots, L, \ k = 1, \ldots, D_\ell\}$ are exponentially distributed. The constant $\eta$, has been set equal to zero. Range and azimuth measurements are affected by errors, which are independent, zero mean, Gaussian random variables, with standard deviations $\sigma_r = 20$ m and $\sigma_\theta = 0.5^\circ$, respectively, independent of the SDR. Targets follow a constant acceleration model (commonly used to account maneuvers [48]), wherein initial position, initial velocity, and acceleration are randomly generated at each run. The scan period is 1 s, and the search area is $\pm 60^\circ$ and 40 to 140 km, where $K = 20$ may be present, and this value is unknown. With reference to Fig. 4, the radar is set so as to detect targets with velocities up to $v_{max}$, specified in each experiment, and accelerations up to $a_{max} = 20$ m/s$^2$. The maximum number of consecutive misses in the Track Formation stage has been set to $P = 4$ and the minimum number of plots required by the Track Pruning stage in each trajectory to $Q = 3$, which is the minimum value to guarantee that the acceleration constraint can be checked. As to the Track Smoothing stage, a standard linear regression is adopted.

In the first set of figures, $v_{max}$ is 100 m/s. Fig. 5 reports $PD_{in}$ and $PD_{out}$ versus $FAR_{in}$ for various SDR’s when $L = 1$ and $L = 10$ scans are integrated, while Fig. 6 shows the corresponding RMSE on the estimated target position. Recall that, for $L = 1$, the scheme in Fig. 2 reduces to the traditional detector in Fig. 1. When $L = 10$, virtually all the detection gain obtained by lowering the threshold $\gamma_1$ in the first stage is maintained for SDR $\geq 12$ dB. When SDR $= 9.30$ dB most of this gain is still preserved: notice in particular that, lowering...
\[ \gamma_1 \] so as to have \( \text{FAR}_{\text{in}} = 1700 \) per minute, a detection gain of 100% is possible with respect to the traditional detection scheme in Fig. 1, boosting PD from 0.2 to 0.4. As to the RMSE, a larger accuracy in the position measurements is possible at the output of the second stage for all the considered interval of \( \text{FAR}_{\text{in}} \), and this accuracy improves as \( \text{FAR}_{\text{in}} \) is increased. Observe that, when \( \text{FAR}_{\text{in}} = 1 \) per minute, \( \text{PD}_{\text{out}} = \text{PD}_{\text{in}} \), since the second stage cannot confirm more plots than those present in the input plot-list; however, \( \text{RMSE}_{\text{out}} \) is lower than \( \text{RMSE}_{\text{in}} \), as the track information can be exploited to refine position estimates.

Fig. 7 shows PD\(_{\text{in}}\) and PD\(_{\text{out}}\) versus \( L \) for different values of SDR when \( \text{FAR}_{\text{in}} = 10^3 \) per min, and Fig. 8 reports the corresponding RMSE on the estimated target position. It can be seen that the PD\(_{\text{out}}\) rapidly reaches its maximum value, and that for \( L = 10 \) (which corresponds to an observation window of 10 s) PD\(_{\text{out}}\) saturates in all the considered SDR’s. As to \( \text{RMSE}_{\text{out}} \), instead, it decreases monotonically, since longer trajectories give rise to position estimates with smaller errors.

Observe that both PD\(_{\text{out}}\) and RMSE\(_{\text{out}}\) remain equal to their corresponding input values for \( L < 3 \); this happens since the Track Pruning stage in Fig. 4 shrinks the trajectories with less than \( Q = 3 \) alarms to the final plot, so that the second stage does not act if less than 3 scans are integrated.

Fig. 9 shows PD\(_{\text{out}}\) versus FAR\(_{\text{out}}\)—i.e., the receiver operating characteristic (ROC)—for SDR = 12.0 dB when \( L = 1 \) and \( L = 10 \) scans are integrated. FAR\(_{\text{out}}\) is reported in logarithmic scale, and different values of \( \text{FAR}_{\text{in}} \) are considered. Observe that, when \( L = 10 \), the PD\(_{\text{out}}\) curve for a given FAR\(_{\text{in}}\) level ends when \( \text{FAR}_{\text{out}} = \text{FAR}_{\text{in}} \), since this is the largest admissible FAR. Moreover, these curves have a smaller slope than PD for the traditional scheme \( (L = 1) \), which means that the proposed two-stage scheme is less sensitive to threshold’s variations: to give an example, in the range \( 1 \leq \text{FAR} \leq 10^3 \), while PD of the traditional scheme increase from 0.35 to 0.61, PD\(_{\text{out}}\) for \( L = 10 \) is always around 0.6 for \( \text{FAR}_{\text{in}} = 10^3 \). It is also interesting to notice that, when \( \text{FAR}_{\text{in}} \) varies, the ROC curves of the proposed two-stage scheme describe a region, which lies above the ROC curve of the traditional scheme: therefore, a better performance can be achieved and a larger freedom is given at the system design stage.

In the second set of figures, \( v_{\text{max}} \) is increased up to approximately Mach-2. Figs. 10 and 11 show PD and RMSE versus \( \text{FAR}_{\text{in}} \), respectively, for different values of SDR when \( L = 1 \) and \( L = 10 \) scans are integrated, and \( v_{\text{max}} = 300 \) m/s, while Figs. 12 and 13 consider the same scene when \( v_{\text{max}} = 600 \) m/s. A comparison with Figs. 5 and 6 shows that almost no loss is incurred when \( \text{FAR}_{\text{in}} \leq 10^3 \), so that large gains with respect to the traditional scheme \( (L = 1) \) are possible even in the detection of fast targets. When \( \text{FAR}_{\text{in}} \) is increased to values larger than \( 10^3 \), the detection gain remain almost unchanged, but the accuracy in the position estimation degrades, and, when \( v_{\text{max}} = 600 \) m/s, it becomes poorer than that of the traditional scheme. We remark, however, that these values of FAR are not of interest, as the computational complexity needed to run Algorithm 1 becomes comparable to that of the Viterbi algorithm and, then, unaffordable.

In Fig. 14 we report a short data snapshot of 120 scans (i.e., 2 minutes) for SDR = 12.0 dB, \( \text{FAR}_{\text{in}} = 10^3 \) per min, and \( v_{\text{max}} = 600 \) m/s. The black dots represent the alarm’s
positions, while the red straight lines indicate the true target trajectories. Plot (a) reports the input of the second stage, while Plots (b) and (c) refer to the outputs when $L = 1$ and $L = 10$, respectively. It is seen that the proposed two-stage scheme is able to confirm 1030 alarms, as opposite to the 619 alarms of the traditional scheme, and all these alarms are correctly located along the true target trajectories.

Finally, for comparison purposes, we analyze the case where the primary threshold $\gamma_1 \rightarrow -\infty$, and the detection architecture in Fig. 2 reduces to the detectors operating on raw input data considered in [2]–[4], [25]. To limit the computational burden, we restrict the search area to $\pm 3^\circ$ and 98 to 102 km, where one target with velocity up to $v_{\text{max}} = 100$ m/s is simulated. The scan period is 1 s, FAR$_{\text{in}} = 1$ per minute, and all other parameters remain unchanged.

Fig. 15 reports PD$_{\text{out}}$ versus SDR for different values of FAR$_{\text{in}}$ when $L = 1$ and $L = 10$ scans are integrated, while Fig. 16 shows the corresponding RMSE’s on the estimated target position. Similarly to the previous plots, PD$_{\text{out}}$ increases monotonically with FAR$_{\text{in}}$, showing here a gap of about 10 dB at PD$_{\text{out}} = 0.5$ between the traditional scheme in Fig. 1 ($L = 1$) and MFD with raw data ($L = 10$ and $\gamma_1 = -\infty$). Any point between these two extrema can be achieved with proper selection of $\gamma_1$ in the proposed scheme in Fig. 2, and complexity is traded for performance. As to RMSE$_{\text{out}}$, it is decreasing with SDR, but no strict ordering can be observed among different values of FAR$_{\text{in}}$. Furthermore, it can be noticed that in the small SDR regime the detection gain with respect to the traditional scheme in Fig. 1 comes at the price of worse accuracy in the position estimation.

In Figs. 17 and 18, instead, we analyze the performance of the procedure in terms of track estimation and range-rate estimation, respectively, and RMSE$_{\text{out}}$ is reported versus SDR for different values of FAR$_{\text{in}}$ and for $L = 10$. Again, there is no strict ordering among the different levels of FAR$_{\text{in}}$, but it can be observed that FAR$_{\text{in}} = 10$ (which is equivalent to FAR$_{\text{in}} = 5 \cdot 10^1$ in the $\pm 60^\circ$ and 40 to 140 km scenario) delivers, with negligible computational complexity increase with respect to the traditional scheme, a gain in the detection probability, a higher accuracy in the position estimation over a wide range of SDR’s (corresponding to PD$_{\text{out}} \geq 5 \cdot 10^{-2}$), and a range-rate information (otherwise not available).

V. CONCLUSION

In this work we have proposed and analyzed a two-stage architecture for target detection in radar systems, wherein a TBD processor operates on a set of candidate plots provided by the Detector and Plot-Extractor. A novel dynamic programming algorithm, which does not require a discretization of the state space, has been derived for plot validation. The complexity analysis reported in Sec. III-E has shown that in standard radar scenarios the proposed algorithm has a computational complexity which can be much lower than that of a multi-frame detection procedure based on the Viterbi algorithm (hardly amenable to a real-time implementation when the

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**Fig. 8.** Root mean square error on the estimated target position at the input and output of the second stage versus the number of integrates scans for different values of SDR. The search area is $\pm 60^\circ$ and 40 to 140 km.

**Fig. 9.** Probability of detection versus false alarm rate at the output of the second stage for different values of FAR$_{\text{in}}$ when SDR = 12.0 dB. The search area is $\pm 60^\circ$ and 40 to 140 km.
number of resolution elements is large). Additionally, the numerical analysis in Sec. IV has shown that the proposed two-stage detection procedure guarantees a large detection gain with respect to the standard radar processing operating at the same level of false alarm rate.

APPENDIX

Let $f_0$ and $f_1$ be the densities of $A_{k,\ell}^2/N_{k,\ell}$ whenever $s_{k,\ell}$ is a false or a true alarm, respectively, and denote $\Lambda = \ln(f_1/f_0)$. Assume that $A_{k,\ell}^2/N_{k,\ell} \geq \gamma_1$, as a result of the first stage thresholding, and that $0 \leq \eta < \gamma_1$. Then the density of $z_{k,\ell}$ in (2) is\footnote{It is a mixed (discrete/continuous) random variables, and the density is computed with respect to the measure defined as the sum of the Dirac measure centered in $\eta$ and the Lebesgue measure.}

\[
g_1(z) = (1 - \text{PD}_{\text{in}}) \mathbb{I}_{\{z=0\}} + \text{PD}_{\text{in}} f_1(z) \mathbb{I}_{\{z \geq \gamma_1\}}
g_0(z) = (1 - \text{PFA}_{\text{in}}) \mathbb{I}_{\{z=\eta\}} + \text{PFA}_{\text{in}} f_0(z) \mathbb{I}_{\{z \geq \gamma_1\}}
\]

whenever it is a true alarm or not, respectively, where $\mathbb{I}_{\mathcal{B}}$ is the indicator function of the event $\mathcal{B}$. Let $\nu$ be the trajectory of the target, and assume scan-to-scan independence conditioned on the absence or presence of a target with this trajectory, then the log-likelihood ratio of $\{z_{\nu,\ell}\}_{\ell=1}^L$ is

\[
\sum_{\ell=1}^L \left[ \left( \ln \frac{\text{PD}_{\text{in}}}{\text{PFA}_{\text{in}}} + \Lambda(z_{\nu,\ell}) \right) \mathbb{I}_{\{\nu_{\ell} \neq 0\}} + \ln \frac{1 - \text{PD}_{\text{in}}}{1 - \text{PFA}_{\text{in}}} \mathbb{I}_{\{\nu_{\ell} = 0\}} \right] = \sum_{\ell=1}^L \left[ \Lambda(z_{\nu,\ell}) \mathbb{I}_{\{\nu_{\ell} \neq 0\}} + \kappa \mathbb{I}_{\{\nu_{\ell} = 0\}} \right] + L \ln \frac{\text{PD}_{\text{in}}}{\text{PFA}_{\text{in}}}
\]

where $\kappa = \ln \frac{1 - \text{PD}_{\text{in}}}{1 - \text{PFA}_{\text{in}}}$. The uncertainty so as to $\nu$ can be removed by maximizing over the set of admissible trajectories $T_{k,\ell}$, and the generalized likelihood ratio test (see [49]) amounts to compare with a threshold the statistic

\[
\max_{\nu \in \mathcal{R}_{k,\ell}} \sum_{\ell=1}^L \left[ \Lambda(A_{k,\ell}^2/N_{k,\ell}) \mathbb{I}_{\{\nu_{\ell} \neq 0\}} + \kappa \mathbb{I}_{\{\nu_{\ell} = 0\}} \right]
\]

(7)

where it has been exploited the fact that $z_{\nu_{\ell},\ell} = A_{k,\ell}^2/N_{k,\ell}$ if $\nu_{\ell} \neq 0$.

Observe that if $\Lambda$ is an affine, increasing function, i.e., $\Lambda(z) = az + b$, $a > 0$, then (7) is equivalent to

\[
\max_{\nu \in \mathcal{R}_{k,\ell}} \sum_{\ell=1}^L \left( A_{k,\ell}^2/N_{k,\ell} \mathbb{I}_{\{\nu_{\ell} \neq 0\}} + \kappa' \mathbb{I}_{\{\nu_{\ell} = 0\}} \right)
\]

which corresponds to (3) if $\eta$ is set equal to $\kappa' = (\kappa - b)/a$. E.g., if $f_0$ and $f_1$ are densities of exponential distributions with parameters $\lambda_0$ and $\lambda_1$, respectively, with $\lambda_0 > \lambda_1$ (this
is a common model for radar measurements), then $\Lambda$ is affine and increasing.

**REFERENCES**


Fig. 12. Detection probability at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.

Fig. 13. Root mean square error on the estimated target position at the input and output of the second stage versus the input false alarm rate for different values of SDR. The search area is ±60° and 40 to 140 km.


v_\text{max} = 100 \text{ m/s}, a_\text{max} = 20 \text{ m/s}^2, \text{FAR}_{\text{out}} = 1 \text{ per min}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure15}
\caption{Detection probability at the output of the second stage versus SDR for different values of \text{FAR}_{\text{out}}. The search area is \pm 3^\circ \text{ and 98 to 102 km.}}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure16}
\caption{Root mean square error on the estimated target position at the output of the second stage versus SDR for different values of \text{FAR}_{\text{in}}. The search area is \pm 3^\circ \text{ and 98 to 102 km.}}
\end{figure}

**Emanuele Grossi** (M’08) was born in Sora, Italy on May 10, 1978. He received with honors the Dr. Eng. degree in Telecommunication Engineering in 2002 and the Ph.D. degree in Electrical Engineering in 2006, both from the University of Cassino, Italy. In 2005 he was visiting Scholar with the Department of Electrical & Computer Engineering of the University of British Columbia, Canada, and in 2009 he had a visiting appointment in the Digital Technology Center, University of Minnesota, MN. Since February 2006, he is assistant professor at the University of Cassino. His research interests concern wireless communication systems, radar detection and tracking, and statistical decision problems with emphasis on sequential analysis.

**Luca Venturino** (S’03-M’06) received with honors the Dr. Eng. degree in Telecommunication Engineering in 2002, and the Ph.D. degree in Electrical Engineering in 2006, both from the Università degli Studi di Cassino, Italy. Luca Venturino is now an Assistant Professor with the Department of Electrical and Information Engineering at the Università degli Studi di Cassino e Lazio Meridionale, Italy, engaged in research and teaching activities on telecommunications. His research interests concern resource allocation in cellular networks, space-time signal processing in wireless multiple-input multiple-output systems, and signal detection in radar. In 2004 and 2009, he was Visiting Scholar with the Department of Electrical Engineering at the Columbia University, New York, NY. Between 2006 and 2008, he spent eight months at NEC Laboratories America, Princeton, NJ, as Research Associate.

**Marco Lops** (M’96-SM’01) was born in Naples (Italy) on March, 16 1961. He obtained his “Laurea” and his Ph. D. degrees from “Federico II” University (Naples), where he was assistant (1989-1991) and associate (1991-2000) professor. Since March 2000 he has been a professor at University of Cassino and, in 2009-2011, he was also with ENSEEIHT (Toulouse). In fall 2008 he was a visiting professor with University of Minnesota and in spring 2009 at Columbia University. His research interests are in Detection and Estimation, with emphasis on Communications and Radar Signal Processing.
Fig. 17. Root mean square error on the estimated target trajectory at the output of the second stage versus SDR for different values of FAR$_{in}$. The search area is ±3$^\circ$ and 98 to 102 km.

Fig. 18. Root mean square error on the estimated target range-rate at the output of the second stage versus SDR for different values of FAR$_{in}$. The search area is ±3$^\circ$ and 98 to 102 km.