Detection rate optimization for Swerling targets in Gaussian noise

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Abstract—In this work, we consider a pulse radar and study the tradeoff between integration time and scan rate for diverse target scattering models. At the design stage, we optimize the available degrees of freedom (namely, pulse train length and detection threshold) so as to maximize the detection rate, defined as the average number of detections from the target per unit of time, subject to a constraint on the false alarm rate, which is the average number of false alarm from the monitored area per unit of time. This objective function allows to carefully balance the contrasting needs for a large probability of detection (achievable through a large dwell time) and a short scan time. Closed-form solutions are provided for Swerling’s Case 1 and 3 target fluctuation and for the Marcum non-fluctuating model, while, for the Swerling’s Case 2 and 4, the solution is found with the aid of computer simulation. A thorough performance analysis is given to show the achievable tradeoffs among the principal system parameters under the different target models.

Index Terms—Detection rate, false alarm rate, pulse radar, radar detection, scan time, surveillance, Swerling targets.

I. INTRODUCTION

Scanning radars monitor a given portion of the sky in order to detect the presence of targets for timely threat prevention. In these systems, the probability of detection can be increased, for fixed transmit power and probability of false alarm, by enlarging the dwell time in each angular direction, so as to increase the time-on-target (TOT) and then the amount of integrated energy at the receiver; this, however, has the drawback of increasing the scan duration and then the reaction time of the radar [1]. To account for such contrasting needs, the cumulative probability of detection, defined as the probability of detecting the target at least once in a preassigned time interval where multiple scans take place, has been proposed as a figure of merit for system optimization, since it can measure the capability of a search radar to detect a new target within a preassigned time interval [2]–[4] or, if the target is closing, before it reaches a given range [5]–[7].

This concept has been recently put forward in [8]–[12]. Following a Neyman-Pearson approach, where the probability of detection is maximized under a constraint on the probability of false alarm, the objective function considered therein is the detection rate (DR), defined as the average number of detections per unit of time from the target, and a constraint is imposed on the false alarm rate (FAR), defined as the average number of false alarms per unit of time from the inspected area. The adoption of FAR (and the closely related false alarm time, the mean time elapsing between two consecutive false alarms [13]–[15]) is quite common in radar applications, for it has a direct impact on the computational requirements for real-time data processing and on the capacity of the human operator to take actions by monitoring the hits visualized on the radar scope [16], [17]. On the other hand, under the hypothesis of target presence, the natural counterpart of FAR is DR, which, being equal to the ratio of the probability of detection to the scan time (see next), is by definition able to jointly account for these two key parameters. Furthermore, as shown in [10], DR is also equal to the inverse of the average value of the detection time (defined as the random time elapsing before a new detection from a prospective target arrives) and is closely related to the quantile of the detection time and to the cumulative probability of detection (namely, they can be bounded by a decreasing function of DR). Hence, on top of being desirable by itself, a large DR is beneficial, as it implies a short time interval between consecutive detections from a persistent target or a fast detection of a new target. Also, a large DR may facilitate subsequent track-before-detect [18]–[25] and/or tracking [26]–[29] algorithms, which follow the detector in the radar processing chain, since more frequent hits result in a smaller association gate in the track estimation process.

In this work, we continue along the path marked out by [10], and we tackle the problem of system optimization in pulse radars by maximizing DR under a constraint on FAR for various target fluctuation models. Starting from the preliminary results in [30],1 we consider here the following radar fluctuation models [13], [14]: Swerling’s Case 1, Swerling’s Case 3, and Marcum’s non fluctuating model (here called Case 0), where a coherent integrator can be used, and Swerling’s Case 2 and 4, where an incoherent integrator is used. We show that the optimization problem takes slightly different forms in these two scenarios, and, for Cases 1, 3, and 0, a closed-form solution for the optimum length of the pulse train is given, while, for Cases 2 and 4, the solution is found with the aid of computer simulations. Finally, a thorough performance analysis is provided to show the impact of the system parameters over the optimum DR and highlight the

1Notice that in the conference paper [30] only a coherent radar is considered, the main results are reported without proofs, and a limited performance analysis is given.
corresponding tradeoffs between the probability of detection and the scan time.

The remainder of the paper is organized as follows. In the next section, the target models and the considered radar systems are presented. In Sec. III, the optimization problem is tackled, while the performance analysis is provided in Sec. IV. Finally, concluding remarks are given in Sec. V.

For the reader’s sake, the list of the most frequently used symbols is provided next.

- \( K \): Number of range bins
- \( M \): Number of azimuth bins
- \( N \): Pulse train length
- \( T \): Pulse repetition time (PRT)
- \( T_s \): Scan time
- \( E \): Received energy per pulse from the target
- \( \sigma^2 \): Noise variance
- \( \rho \): SNR per pulse
- \( \gamma \): Detection threshold
- \( P_d \): Probability of detection
- \( P_{fa} \): Probability of false alarm
- \( N_{\text{min}} \): Minimum pulse train length
- \( N_{\text{max}} \): Maximum pulse train length
- \( \text{DR} \): Detection rate
- \( \eta \): Maximum tolerable FAR level

\[ A^2_n = A^2 \] is a unit-mean gamma random variables with variance \( \frac{1}{2} \), and \( \theta_1 = \cdots = \theta_n = \theta \);

- Swerling’s Case 4 (Chi-square with four degrees of freedom and pulse-to-pulse fluctuation): \( \{A^2_n\}_{n=1}^N \) are i.i.d. unit-mean gamma random variables with variance \( \frac{1}{2} \), and \( \{\theta_n\}_{n=1}^N \) are i.i.d.;

- Marcum’s non-fluctuating model, here called Case 0: \( A^2_1 = \cdots = A^2_n = 1 \), and \( \theta_1 = \cdots = \theta_n = \theta \).

In the Swerling’s Case 2 and 4, the received signal from the range-azimuth cell under inspection at the \( n \)-th pulse is

\[
r_n = \begin{cases} 
    A_n e^{2\pi i \theta_n} \sqrt{E} + w_n, & \text{under } H_1 \\
    w_n, & \text{under } H_0
  \end{cases}
\]

where: \( E \) is the received energy per pulse from the target; \( w_n \sim \mathcal{N}_c(0, \sigma^2) \), a complex circularly-symmetric Gaussian random variable with variance \( \sigma^2 \), is the noise at the \( n \)-th pulse; \( H_0 \) is the null hypothesis (the cell under test contains only noise); and \( H_1 \) is the alternative hypothesis (a target is present in the cell under test). A square-law detector is commonly used,

\[
\frac{1}{\sigma^2} \sum_{n=1}^N |r_n|^2 \geq \frac{H_1}{H_0} \quad \gamma
\]

where \( \gamma \) is the detection threshold. This is the likelihood ratio test (LRT) for the Swerling’s Case 2, while it is an approximation of the LRT in the regions

\[
\frac{\rho - \rho |r_n|^2}{2 + \rho \sigma^2} \ll 1 \quad \text{and} \quad \frac{\rho |r_n|^2}{2 + \rho \sigma^2} \gg 1
\]

for the Swerling’s Case 4, \( \rho = E/\sigma^2 \) denoting the signal-to-noise ratio (SNR) per pulse. The probability of false alarm and detection are \([31], [32]\)

\[
P_{fa} = Q(N, \gamma) = e^{-\gamma} \sum_{n=0}^{N-1} \frac{\gamma^n}{n!}, \quad \text{for Case 2}
\]

\[
P_d = \begin{cases} 
    1 - \frac{1}{(1+\gamma)^N} \sum_{k=0}^{N} \binom{N}{k} \left( \frac{\theta}{2} \right)^{N-k} e^{-\frac{1}{2} \gamma}, & \text{for Case 4}
  \end{cases}
\]

where \( Q(\cdot, \cdot) \) denotes the regularized upper incomplete gamma function.

In the Swerling’s Case 1 and 3 and in Marcum’s Case 0, instead, the target response does not fluctuate during the pulse train, and a coherent elaboration of the \( N \) pulses is possible. Standard Doppler processing, which defines \( N \) Doppler bins \([33]\), amounts to summing up the received data samples after coherent demodulation. Neglecting the straddle loss, the signal from the range-Doppler-azimuth cell under inspection can be written as

\[
\frac{1}{\sqrt{N}} \sum_{n=1}^N r_n = \begin{cases} 
    A_n e^{2\pi i \theta_n} \sqrt{N E} + w, & \text{under } H_1 \\
    w, & \text{under } H_0
  \end{cases}
\]

where \( w \sim \mathcal{N}_c(0, \sigma^2) \). The coherent detector for the cell under test is, therefore,

\[
\frac{1}{N \sigma^2} \sum_{n=1}^N |r_n|^2 \geq \frac{H_1}{H_0} \gamma
\]
which is the LRT for all these cases. The probability of false alarm and detection are [31], [32]

\[ P_{fa} = e^{-\gamma} \quad (7a) \]

\[ P_{d} = \left\{ \begin{array}{ll} e^{-\gamma N_{\rho}}, & \text{for Case 1} \\ \left(1 + \frac{\gamma N_{\rho}}{(1 + \frac{\gamma N_{\rho}}{2})} \right) e^{-\gamma N_{\rho}}, & \text{for Case 3} \\ Q_{1}\left(\sqrt{2N_{\rho}}, \sqrt{2\gamma}\right), & \text{for Case 0} \end{array} \right. \quad (7b) \]

respectively, where \( Q_{m}(\cdot, \cdot) \) is the generalized Marcum \( Q \)-function of order \( m > 0 \).

The target models considered here represent the cases where the response is completely correlated (constant) or uncorrelated from pulse to pulse. The probability of detection in the partially correlated case has been found in [34], [35], and, when clutter is also present, in [36]–[39]. Even if it is not considered here, we observe that the results in the partially correlated case will fall between the two extremes of totally correlated and uncorrelated target models, that are, inter alia, of particular importance in many situations of interest. Indeed, often the integration interval is short enough and the target far enough that the target’s response is relatively stationary over the integration interval; on the other hand, if frequency agility is used, so that each pulse has a different carrier frequency, the target’s response is uncorrelated.

### III. System Optimization

The detection performance of radar systems is commonly measured through the probabilities of detection and false alarm. However, these metrics do not incorporate any information about time and number of resolution cells, and radar engineers are more often concerned with FAR (closely related to the false alarm rate), since it has a direct impact on the computational requirements of the radar and on the capacity of the operator in handling the hits produced by the detector. Likewise, DR is a relevant figure of merit, since it accounts for the time elapsing between consecutive detections from the same target: since both the probability of detection \( P_{d} \) and the scan time \( T_{s} \) increase with the dwell time, a marginal increment of \( P_{d} \) may not be worthwhile if it comes at the price of an excessive increment of \( T_{s} \), especially in surveillance operations.

For these reasons, FAR and DR are relevant figure of merit that can be used for system optimization. In this work, we maximize DR over the system parameter (namely, the length of the pulse train \( N \) and the detection threshold \( \gamma \)) under a constraint on the maximum tolerable FAR level. Since the number of resolution cells is \( KMN \) (the product of the number of range, azimuth, and Doppler bins) for a coherent radar (used in Cases 0, 1, and 3) and \( KM \) for an incoherent one (used in Cases 2 and 4), the average number of false alarms in a scan is \( KMN P_{fa} \) and \( KM P_{fa} \), respectively, and

\[ \text{FAR} = \left\{ \begin{array}{ll} \frac{KMN P_{fa}}{T_{s}} = \frac{K}{T_{s}} P_{fa} & \text{for Cases 1, 3, 0} \\ \frac{KM P_{fa}}{T_{s}} = \frac{K}{T_{s}} P_{fa} & \text{for Cases 2, 4} \end{array} \right. \quad (8) \]

Therefore, constraining FAR is equivalent to constraining \( P_{fa} \) only in a coherent radar (where they are proportional to each other through the constant coefficient \( K/T \)). As to DR, instead, the average number of detections from the target in a scan is simply \( P_{d} \) (indeed, the target can either be seen once, with probability \( P_{d} \), or missed, with probability \( 1 - P_{d} \)), so that

\[ \text{DR} = \frac{P_{d}}{T_{s}} = \frac{1}{MT} \frac{P_{d}}{N} \quad (9) \]

for both type of radars. Therefore the maximization of DR may lead to quite different results from those obtained by considering only \( P_{d} \); now, by properly adjusting the pulse train length \( N \), one can balance the contrasting needs for a short scan time and a long TOT (i.e., a large \( P_{d} \)).

Denoting \( \eta > 0 \) the maximum tolerable FAR level, and letting \( N_{min} \) and \( N_{max} \), \( N_{min} < N_{max} \), the minimum and maximum lengths of the pulse train, the optimization problem considered here is

\[ \max_{N \in \mathbb{N}, \gamma \in \mathbb{R}} \text{DR} \]

\[ \text{s.t.} \quad \text{FAR} \leq \eta \]

\[ N_{\min} \leq N \leq N_{\max} \quad (10) \]

Since, from (8), \( \text{FAR} \in [0, K/T] \) for a coherent radar (used with Cases 0, 1, and 3) and \( \text{FAR} \in [0, K/(N_{min}T)] \) for an incoherent one (used with Cases 2 and 4), we will always assume that \( \eta \geq K/(N_{min}T) \) for the latter and \( \eta \geq K/T \) for the former: if this is not the case, the constraint on FAR is not active, and we have the trivial solution \( \gamma \leq 0 \) (so that \( P_{d} = 1 \)) and \( N = N_{\min} \), which gives the largest possible DR, \( 1/(MN_{\min}T) \). In the following two sections, Problem (10) is tackled under the target fluctuation models introduced in Sec. II, and, in Sec. III-C, a geometric interpretation of the problem and of its solution is given.

### A. Cases 1, 3 and 0

From Eqs. (8) and (9), Problem (10) can be rewritten as

\[ \max_{N \in \mathbb{N}, \gamma \in \mathbb{R}} \left[ \frac{1}{N} \frac{P_{d}}{T_{s}} \right] \]

\[ \text{s.t.} \quad \frac{1}{N} \frac{P_{fa}}{T_{s}} \leq \alpha \]

\[ N_{\min} \leq N \leq N_{\max} \quad (11) \]

where \( \alpha = \eta T/K \in [0, 1] \) constrains the probability of false alarm. Since, from (7b), \( P_{d} \) is decreasing with \( \gamma \), the smallest threshold satisfying the constraint must be chosen, i.e., from (7a), \( \gamma = -\ln \alpha \), so that \( P_{fa} = \alpha \). The optimum pulse train length can therefore be found as

\[ \arg \max_{N \in \{N_{\min}, \ldots, N_{\max}\}} \left[ \frac{1}{N} P_{d} \right]_{\gamma = -\ln \alpha} \quad (12) \]

The upper limit, \( N_{\max} \), is tied to the target’s velocity and acceleration and is the largest value of the pulse train length for which there is no range and/or azimuth and/or (for a coherent radar) Doppler migration during the processing interval.
The solution to (12) is

\[
N^* = \begin{cases} 
N_{\text{min}}, & \text{if } \alpha \geq e^{-\bar{\gamma}}, \text{ or if } \alpha < e^{-\bar{\gamma}} \text{ and } \frac{x_\alpha}{\rho} \leq N_{\text{min}} \\
\arg \max_{N \in \{N_{\text{min}}, N_{\text{max}}\}} \frac{1}{N} P_d |_{\gamma = -\ln \alpha^*}, & \text{if } \alpha < e^{-\bar{\gamma}} \text{ and } \frac{x_\alpha}{\rho} \geq N_{\text{max}} \\
\arg \max_{N \in \{N_{\text{min}}, \lceil \frac{x_\alpha}{\rho} \rceil, \lceil \frac{x_{\alpha}}{\rho} \rceil \rceil} \frac{1}{N} P_d |_{\gamma = -\ln \alpha^*}, & \text{otherwise}
\end{cases}
\]

(13)

where \(\bar{\gamma}\) is the unique positive value for which the equation

\[
\begin{cases} 
x^2 - (\gamma - 2)x + 1 = 0, & \text{for Case 1} \\
\gamma^2 \left(1 + \gamma \frac{2}{\gamma} - (\frac{2}{\gamma})^2\right) - (1 + \frac{2}{\gamma})^4 = 0, & \text{for Case 3} \\
x \cdot Q_2(\sqrt{2x}, \sqrt{2\gamma}) - (1 + x) Q_1(\sqrt{2x}, \sqrt{2\gamma}) = 0, & \text{for Case 0}
\end{cases}
\]

(14)

has a single double root for \(x > 0\), while \(x_\alpha\) is the largest of the two positive simple roots of the same equation when \(\gamma > \bar{\gamma}\).

The proof for Case 1 can be found in [10, Case Study V], while those for the other two models are given in the Appendix. As to the constants \(\bar{\gamma}\) and \(x_\alpha\), we simply have

\[
\bar{\gamma} = 4
\]

(15)

\[
x_\alpha = \frac{1}{2} \left(-\ln \alpha - 2 + \sqrt{\ln^2 \alpha + 4 \ln \alpha}\right)
\]

(16)

for Case 1, while, for the other two cases, no closed-form expression seems to be available. Nevertheless, they can readily be found through numerical methods, and

\[
\bar{\gamma} \approx \begin{cases} 
3.77, & \text{for Case 3} \\
3.39, & \text{for Case 0}
\end{cases}
\]

(17)

while \(x_\alpha\) is reported in Fig. 1, also in comparison to the value in (16).

The following remarks can now be given. Looking at (13), we see that, for probabilities of false alarm above the threshold value \(e^{-\bar{\gamma}}\), DR is maximized with \(N_{\text{min}}\) pulses, independent of the SNR per pulse \(\rho\). Below this threshold, instead, the optimum number of pulses depends on the pair \((\alpha, \rho)\). In the range of values of \((\alpha, \rho)\) not requiring \(N_{\text{min}}\) or \(N_{\text{max}}\) pulses, \(N^*\) provides a cumulated (i.e., after coherent integration) SNR, \(N^* \rho\), approximately equal to \(x_\alpha\) (cfr. Fig. 1), and the pair \((N^*, \gamma^*)\) provides a probability of detection approximately equal to

\[
P_d |_{\gamma = -\ln \alpha, N^* \rho = x_\alpha} = \begin{cases} 
c - \frac{1}{1 + \sqrt{1 + 2 \ln \alpha^*}} - \frac{1}{\ln \alpha^*}, & \text{for Case 1} \\
\left(1 + \frac{2}{\sqrt{2 \lambda}} \ln \alpha\right)^{\frac{1}{\sqrt{2 \lambda}}} - \frac{1}{1 + \sqrt{1 + 2 \ln \alpha^*}}, & \text{for Case 3} \\
Q_1(\sqrt{2x_\alpha}, \sqrt{2\gamma})^{-2 \ln \alpha}, & \text{for Case 0}
\end{cases}
\]

(18)

which depends on \(\alpha\) only (i.e., it is independent of \(\rho\)). This value is shown in Fig. 2 as a function of \(\alpha\): it can be seen that it is approximately flat in a wide range of commonly required levels of probability of false alarm and surprisingly low for Cases 1 and 3. Also, in the region where \(N^* \neq N_{\text{min}}, N_{\text{max}}\),

\[
T_s \approx \frac{MT x_\alpha}{\rho}
\]

(19a)

\[
\text{DR} \approx \rho P_d |_{N^* \rho = x_\alpha, \gamma = -\ln \alpha} = \frac{1}{MT x_\alpha}
\]

(19b)

i.e., the scan time is approximately inversely proportional to the SNR per pulse, and, as a consequence of the fact that \(P_d\) is approximately constant, DR grows linearly with the SNR per pulse. Finally, we observe that the optimum value of DR tends to \(1/(MN_{\text{min}} T)\) when \(\rho \to \infty\) (since \(P_d \to 1\)) and tends to \(\alpha/(MN_{\text{min}} T)\) when \(\rho \to 0\) (since \(P_d \to P_{fa}\)).

B. Cases 2 and 4

From (8) and (9), Problem (10) can be rewritten as

\[
\max_{N \in \mathbb{N}, \gamma \in \mathbb{R}} \frac{1}{N} P_d \\
\text{s.t.} \quad \frac{1}{N} P_{fa} \leq \alpha \\
N_{\text{min}} \leq N \leq N_{\text{max}}
\]

(20)
where $\alpha = \eta T/K \in [0,1]$ constrains the probability of false alarm per sample. From (4b), $P_d$ is decreasing with $\gamma$, so that, for each fixed $N$, the smallest threshold satisfying the constraint must be chosen, i.e., from (4a), $\gamma = Q^{-1}(N, \alpha N)$, where $Q^{-1}(\cdot, \cdot)$ denotes the inverse of the regularized upper incomplete gamma function, so that $\frac{1}{N} P_d = \alpha$. The optimum pulse train length can therefore be computed as

$$N^* = \arg\max_{N \in \{N_{\min}, \ldots, N_{\max}\}} \frac{1}{N} P_d |_{\gamma = Q^{-1}(N, \alpha N)}.$$

Unfortunately, there appears to be no closed-form solution for the problem in (21)—which, for Case 2, has been studied in [10, Case Study III]—, so that $N^*$ has to be evaluated numerically. In Fig. 3, we report the objective function in (21) versus the pulse train length $N$ for different values of $\alpha$ and $\rho$. It can be seen that, for each value of $\rho$, the objective function is increasing and then decreasing with $N$ when $\alpha$ is larger than a certain threshold value, say $\alpha_{\rho}$, exhibiting a maximum at $N = n_{\alpha_{\rho}}$, while it is monotonically decreasing with $N$ for $\alpha < \alpha_{\rho}$. In Fig. 4, $\alpha_{\rho}$ is plotted versus $\rho$, and it can be seen that it is decreasing, consistent with Fig. 3. From this analysis, we therefore conjecture (and the numerical results in Sec. IV confirm) that the solution of (21) is

$$N^* = \begin{cases} N_{\min}, & \text{if } \alpha \geq \alpha_{\rho}, \text{ or if } \alpha < \alpha_{\rho} \text{ and } n_{\rho,\alpha} \leq N_{\min} \\ N_{\max}, & \text{if } \alpha < \alpha_{\rho} \text{ and } n_{\rho,\alpha} \geq N_{\max} \\ n_{\rho,\alpha}, & \text{otherwise} \end{cases} \quad (22a)$$

$$\gamma^* = Q^{-1}(N^*, \alpha N^*). \quad (22b)$$

As in Cases 2, 3, and 0, the optimum value of DR admits a closed-form expression in the small and high SNR limit, and it is equal to $1/(MN_{\min} T)$ when $\rho \to \infty$ (since $P_d \to 1$) and to $\alpha/(MT)$ when $\rho \to 0$ (since $P_d \to P_h$).

C. A geometric interpretation

It has been seen that the optimization problem can be cast as in (11) and (20) according to the target fluctuation model. In both cases, since $P_d$ is increasing with $P_h$, the threshold $\gamma$ corresponding to the largest value of $P_h$ should be chosen for each fixed $N$. Denoting $\gamma_{\alpha, N}$ such a threshold, the optimum pulse train length can be found by solving

$$N^* = \arg\max_{N \in \{N_{\min}, \ldots, N_{\max}\}} \frac{1}{N} P_d(N) \quad (23)$$

where, with a slight abuse of notation, we have denoted $P_d(N)$ as the detection probability when the pulse train length is $N$ and the threshold is $\gamma_{\alpha, N}$.

Observe that (23), which subsumes (12) and (21) as special cases, holds independently of the noise and/or target fluctuation models, here concealed in the expressions of $P_d$ and $\gamma_{\alpha, N}$. This high-level representation allows us to give an interesting geometrical interpretation of the problem. Indeed, if $P_d(N)$ is plotted versus $N$, we can observe that $P_d(N)$ is increasing and always larger than $\alpha$ (Cases 1, 3, and 0) and $\alpha N$ (Cases 2, and 4). On this plot, the term $\frac{1}{N} P_d(N)$ is simply the slope of the line segment from the origin to the point $(N, P_d(N))$, so that, in order to maximize DR, we need to select the value of $N$ that results in the line segment with largest slope; see Fig. 5 for an illustrative example.

In particular, if $P_d(N)$ is strictly increasing and $N_{\max} - N_{\min} \geq 2$, the conditions to have a local maximum at $N \in \{N_{\min} + 1, \ldots, N_{\max} - 1\}$ are

$$\begin{cases} \frac{1}{N} P_d(N) > \frac{1}{N-1} P_d(N - 1) \\ \frac{1}{N} P_d(N) > \frac{1}{N+1} P_d(N + 1) \end{cases} \quad (24)$$

The parameter $\gamma_{\alpha, N}$ is such that $P_h = \alpha$ for Cases 1, 3, and 0, or $P_h = \alpha N$ for Cases 2 and 4; if multiple values of the threshold satisfy this equation, the one corresponding to the largest $P_d$ must be chosen.

Here, we are implicitly excluding pathological situations, where the considered detector has a probability of detection smaller than the probability of false alarm (namely, that it performs worse than just guessing) or has a receiver operating characteristic not increasing with $N$. 

![Figure 3. Objective function in (21) versus the number of pulses for different values of $\alpha$ and $\rho$ for Cases 2 and 4. Observe that, from (8), $N \leq P_d/\alpha \leq 1/\alpha$.](image)

![Figure 4. Threshold value of $\alpha$ versus the SNR per pulse for the Cases 2 and 4.](image)
or, equivalently,
\[ P_d(N+1) - P_d(N) < \frac{1}{N} P_d(N) < P_d(N) - P_d(N-1). \] 
(25)

This means that \( \frac{1}{N} P_d(N) \) must be bounded between the slopes of \( P_d(N) \) right after and before the point \( N \), and that \( P_d(N) \) itself must be concave at \( N \).

IV. PERFORMANCE ANALYSIS

We consider a pulse radar with \( K = 2000 \) range bins, \( M = 60 \) azimuth bins, and a pulse repetition time of \( T = 1 \text{ ms} \). For this system, DR is optimized over \((N, \gamma)\) according to Problem (10) when \( N_{\text{min}} = 4 \) and \( N_{\text{max}} = 100 \) (hence, TOT varies between 4 and 100 ms, while \( T_s \) between 0.24 and 6 s). DR and FAR are measured in detections per second (det/s) and false alarms per second (fa/s), respectively.

The first two figures show the optimized DR versus \( \eta \) when \( \rho = -5 \) and 5 dB for Cases 1, 3, and 0 (Fig. 6), and when \( \rho = 0 \) and 5 dB for Cases 2 and 4 (Fig. 7). This is akin to a receiver operating characteristic (ROC) of the system when the time resource is also taken into account, in that the detection rate (instead of the probability of detection) is plotted as a function of the false alarm rate (instead of the probability of false alarm). It can be seen that, clearly, DR is increasing with \( \eta \) and \( \rho \); also, for the inspected values of SNR per pulse, it is decreasing with the variance of the target response: i.e., Case 0 (zero variance) outperforms Case 3 (variance equal to 1/2) which outperforms Case 1 (unit variance), and Case 4 (variance equal to 1/2) outperforms Case 2 (unit variance).

Fig. 8 shows the optimized DR versus \( \rho \) for all the target fluctuation models when \( \eta = 0.03 \text{ fa/s} \) (i.e., 1.8 false alarms per minute). It can be seen that, at higher \( \rho \)'s, DR is larger for target models with a lower variance (i.e., Case 0 is better than Case 3, which is better than Case 1, and Case 4 is better than Case 2), and vice versa at lower \( \rho \)'s. Furthermore, pulse-to-pulse fluctuation (Cases 2 and 4) is more rewarding at higher \( \rho \)'s than scan-to-scan fluctuation (Cases 1 or 3) and vice versa at lower \( \rho \)'s. It is therefore interesting to notice that this behavior, well known when \( P_d \) is maximized with a constraint on \( P_{\text{fa}} \)—since, at lower SNR’s, \( P_d \) benefits from a higher variance of the signal fluctuation, and integration is preferable over diversity, while, at higher SNR’s, it is the other
consistent with (18). The ripple is caused by the mismatch between the optimum value $N^*$, which, in this region, is equal to either $\lceil x_\alpha / \rho \rceil$ or $\lfloor x_\alpha / \rho \rfloor$, and the maximizer of the objective function in the relaxed problem, $x_\alpha / \rho$. Such a ripple is increasing with $\rho$, since $x_\alpha / \rho$ decreases, and the relative mismatch between $N^*$ and $x_\alpha / \rho$ increases. In this region where the required $P_d$ is approximately flat, the required scan time $T_s$ decreases approximately as $1/\rho$, as it can also be observed in Fig. 10, and DR grows linearly with $\rho$ (cfr. Fig. 8), consistent with (19). At lower and higher SNR’s, $N^* = N_{\text{max}}$ ($T_s = 0.24$ s) and $N^* = N_{\text{min}}$ ($T_s = 6$ s), respectively, and $P_d$ saturates towards 0 and 1, respectively. As for Cases 2 and 4, we observe the same behavior at higher and lower SNR’s, while, at intermediate SNR’s ($\rho$ between $-1.8$ and $8$ dB, in the inspected scenario), $P_d$ is slightly decreasing with $\rho$ (instead of being approximately constant as for the other cases) but remains in the range between 0.6 and 0.8.

In general, from Fig. 10, it can be inferred that, the smaller the variance of the target response is, the larger the scan time should be. As to cases Cases 1, 3, and 0, indeed, non-fluctuating targets require the largest value for the scan time.

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**Figure 8.** Optimized DR versus SNR per pulse for different target fluctuation models when $\eta = 0.03 \text{ fa/s}$; the curve corresponding to Case 2 lies above the one corresponding to Case 4 for $\rho \leq 2.78$ dB, and vice versa for $\rho \geq 2.78$ dB.

**Figure 9.** Probability of detection yielding the optimized DR in Fig 8 versus SNR per pulse for different target fluctuation models and $\eta = 0.03 \text{ fa/s}$.

**Figure 10.** Scan time yielding the optimized DR in Fig 8 versus SNR per pulse for different target fluctuation models and $\eta = 0.03 \text{ fa/s}$.

**Table I**

<table>
<thead>
<tr>
<th>$\rho$ [dB]</th>
<th>$\gamma^*$, Cases: 1, 3, and 0</th>
<th>$N^*$, Cases: 1, 3, 0, 2 and 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-10$</td>
<td>18.02 153.8</td>
<td>100 100 100 100</td>
</tr>
<tr>
<td>$-8$</td>
<td>18.02 153.8</td>
<td>100 100 100 100</td>
</tr>
<tr>
<td>$-6$</td>
<td>18.02 153.8</td>
<td>64 80 95 100</td>
</tr>
<tr>
<td>$-4$</td>
<td>18.02 153.8</td>
<td>40 50 60 100</td>
</tr>
<tr>
<td>$-2$</td>
<td>18.02 153.8</td>
<td>25 32 38 100</td>
</tr>
<tr>
<td>0</td>
<td>18.02 95.54</td>
<td>16 20 24 53</td>
</tr>
<tr>
<td>2</td>
<td>18.02 60.79</td>
<td>10 13 15 27</td>
</tr>
<tr>
<td>4</td>
<td>18.02 41.67</td>
<td>6 8 10 14</td>
</tr>
<tr>
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<td>18.02 31.88</td>
<td>4 5 6 8</td>
</tr>
<tr>
<td>8</td>
<td>18.02 24.57</td>
<td>4 4 4 4</td>
</tr>
<tr>
<td>10</td>
<td>18.02 24.57</td>
<td>4 4 4 4</td>
</tr>
</tbody>
</table>

---

way round—, is also observed in the present framework, where DR is maximized with a constraint on FAR.

In the next two figures, we show, as a function of $\rho$, the values of probability of detection (in Fig. 9) and scan time (in Fig. 10) yielding the optimized DR shown in Fig. 8, while, in Table I, the solution (i.e., the optimum values $\gamma^*$ and $N^*$) are reported. From Fig. 9, it can be seen that, for Cases 1, 3, and 0, the required $P_d$ is approximately constant in a wide range of intermediate SNR’s, and

\[
P_d\big|_{N=N^*,\gamma=\gamma^*} \approx \begin{cases} 
0.35, & \text{for Case 1} \\
0.48, & \text{for Case 3} \\
0.84, & \text{for Case 0}
\end{cases} \quad (26)
\]

consistent with (18). The ripple is caused by the mismatch...
and, from (26), the largest probability of detection at which the radar should operate. At the other extreme lays Case 1, characterized by the largest variance of the target response, which require the smallest scan time and an operating $P_d$ that is less than a half of the one requested by non-fluctuating targets. The intuition is that a relatively small dwell time is sufficient to have a relatively large $P_d$ (and, consequently, a relatively large DR) if a favorable realization of the target response is experienced, while, in the presence of a very weak realization, increasing the scan time is not worthy, for it would result in a marginal increase of the probability of detection; better, instead, a smaller dwell time, so as to rapidly move to the next scan, where the target features an independent realization of its response. Case 3 lies between these two extremes. As to Cases 2 and 4, they require the same scan time, which is larger than that needed by Case 0 in coherent radars. This is related to the fact that, since the target response varies from pulse-to-pulse, a larger dwell time is beneficial to reduce the variance of the statistic due to the target fluctuation.

Finally, for comparison purposes, we consider the alternative design approach where the pair $(N, \gamma)$ is set so as to have $P_d$ as close as possible to some desired value, say $\tilde{P}_d$, under the same FAR constraint, i.e.,

$$\min_{N \in \mathbb{N}, \gamma \in \mathbb{R}} |P_d - \tilde{P}_d|$$

s.t. FAR $\leq \eta$

$$N_{\min} \leq N \leq N_{\max}. \quad (27)$$

The loss in terms of SNR per pulse suffered by this design strategy with respect to that in (10) for a certain DR level $x$ is

$$L(x) = \rho_{\text{DR}=x \text{ in (27)}}/\rho_{\text{DR}=x \text{ in (10)}} \quad (28)$$

and is shown in Fig. 11 for all the target fluctuation models and for $\tilde{P}_d \in \{0.2, 0.4, 0.6, 0.8\}$. At higher and lower DR’s (i.e., higher and lower $\rho$’s), there is no loss, since both strategies require the same number of pulses ($N_{\min}$ and $N_{\max}$, respectively). At intermediate values of DR, instead, the SNR loss can be quite heavy and is larger for values of $\tilde{P}_d$ farther from the optimum one represented in Fig. 9.

In practice, however, a wise strategy is to make a robust choice, i.e., to set the system parameters in such a way that the loss in terms of DR for a fixed FAR is as insensitive as possible to the target fluctuation law, which may not even be known. In this context, the curves in Fig. 11 may be helpful, especially considering the two extreme situations of Swerling’s Case 1 and Marcum’s Case 0: indeed, choosing an operating point for the probability of detection of about 0.5, would entail a maximum loss (as compared to the best achievable DR) of about 1 dB.

V. CONCLUSION

In this work, the problem of maximizing the detection rate under a constraint on the false alarm rate has been tackled in radar systems where the observations are corrupted by Gaussian noise and the target response follows one of the well known Swerling models (Cases 1, 2, 3, or 4) or the Marcum’s non-fluctuating model (Case 0). The optimization problem takes slightly different shapes for Cases 1, 3, 0 and for Cases 2, 4: in the former, the LRT requires a coherent integration, and constraining FAR is equivalent to constraining the probability of false alarm, while, in the latter, the LRT (or an approximation thereof) requires an incoherent integration, and constraining FAR amounts to constraining the probability of false alarm divided by the number of integrated pulses.

In this framework, a closed-form expression for the optimum value of the pulse train length has been provided for Cases 1, 3, and 0, while, for Cases 2, and 4, the solution has been evaluated with the aid of computer simulations. The analysis has shown that, similarly to what happens when the probability of detection is optimized, DR is larger for models with a lower variance in the target response at higher SNR’s, the order being reversed at lower SNR’s, and that pulse-to-pulse (scan-to-scan) fluctuation is beneficial at higher (lower) SNR’s. Furthermore, it has been shown that, in a wide range of SNR’s where the optimum value of the pulse train length is
not on the boundaries of the search interval, the probability of detection required to achieve the largest DR is approximately constant (for Cases 1, 3, and 0) and approximately decreasing (for Cases 2 and 4) with the SNR, so that the growth in DR with the SNR is achieved through a sensible reduction of the time-on-target (and, then, of the scan time). Finally, a numerical example has shown that the alternative strategy of selecting the number of processed pulses so as to have a certain design probability of detection may lead to severe losses (in terms of detection rate) if the chosen \( P_\alpha \) is far from the optimum one, while it appears to be a robust choice if the required \( P_\alpha \) is around the value of 0.5, in which case the maximum loss suffered in case of mismatch is in the order of 1 dB.

**APPENDIX**

Here we derive the solution for the optimum pulse train length for Cases 3 and 0, i.e., we prove that (13) solves (12) for these target models.

Let us start with Case 3. From (7b), the problem in (12) becomes

\[
\arg\max_{N \in \{N_{\min}, \ldots, N_{\max}\}} \frac{1}{N} \left( 1 + \frac{N \rho}{(1 + \frac{\rho}{N})^2} \right)^\gamma e^{- \frac{\gamma}{\gamma^*}}. \tag{29}
\]

To solve (29) we relax the problem and introduce the function \( f : (0, \infty) \rightarrow \mathbb{R} \), defined as

\[
f(x) = \frac{\rho}{x} \left( 1 + \frac{\frac{\rho}{x}}{1 + \frac{\rho}{N}} \right)^\gamma e^{- \frac{\gamma}{\gamma^*}}. \tag{30}
\]

This is a positive function with \( \lim_{x \to 0^+} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = 0 \); furthermore, it is continuously differentiable, with derivative

\[
f'(x) = \frac{\rho e^{- \frac{\gamma}{\gamma^*}}}{x^2 \left( 1 + \frac{\rho}{N} \right)^2} \times \left( \frac{x}{\gamma} - \left( \frac{x}{\gamma} + \frac{1}{\gamma} \right)^2 \right) - \left( \frac{1}{\gamma^*} + \frac{\rho}{N} \right).
\]

Therefore, \( f'(x) \geq 0 \) if and only if

\[
h(x) = 1 + \gamma x - \left( \frac{x}{\gamma} \right)^2 \geq \frac{(1 + \frac{\rho}{N})^4}{\gamma^*} = g(x) \tag{32}
\]

The functions \( h \) and \( g \) can easily be studied, so that the region where \( f'(x) \geq 0 \) can readily be found. Specifically, looking at Fig. 12, we see that \( f'(x) \leq 0 \) \( \forall x \), if \( \gamma \leq \bar{\gamma} \); on the other hand, if \( \gamma > \bar{\gamma} \), we have \( f'(x) < 0 \) if \( x \in (0, x_{\gamma,1}) \cup (x_{\gamma,2}, \infty) \), \( f'(x) = 0 \) if \( x \in \{x_{\gamma,1}, x_{\gamma,2}\} \), and \( f'(x) > 0 \) if \( x > x_{\gamma,2} \), where \( \bar{\gamma} \) is the unique value of \( \gamma \) for which the equation \( h(x) = g(x) \) has two coincident roots, and \( x_{\gamma,1} < x_{\gamma,2} \) are the two roots of the same equation when \( \gamma > \bar{\gamma} \).

This implies that, if \( \gamma^* \leq \bar{\gamma} \), i.e., if \( \alpha \geq e^{-\bar{\gamma}} \), the objective function in (29) is decreasing with \( N \) and is therefore maximized for \( N = N_{\min} \). On the other hand, if \( \gamma^* > \bar{\gamma} \), i.e., if \( \alpha < e^{-\bar{\gamma}} \), three cases may occur. Let \( x_{\alpha} = x_{\gamma,2}|_{\gamma = -\ln \alpha} \) be the point at which the corresponding relaxed function \( f(x) \) achieves a relative maximum. Then, if \( N_{\min} \rho \geq x_{\alpha} \), the objective function in (29) is decreasing and, again, is maximized for \( N = N_{\min} \). If \( N_{\max} \rho \leq x_{\alpha} \), the objective function in (29) is either decreasing (if \( N_{\max} \rho \leq x_{\gamma,1}|_{\gamma = -\ln \alpha} \) or decreasing and then increasing (if \( N_{\max} \rho \geq x_{\gamma,1}|_{\gamma = -\ln \alpha} \)), so that it is maximized at an extreme point of the search set, i.e., \( N \in \{N_{\min}, N_{\max}\} \). Finally, if \( N_{\min} \rho < x_{\alpha} < N_{\max} \rho \), the objective function is maximized either at \( \arg\min_{N \in \{N_{\min}, \ldots, N_{\max}\}} N \rho - x_{\alpha} \) or at the boundaries of the set, i.e., \( N \in \{N_{\min}, N_{\max}\} \). This proves (13) for the Case 3.

Let us now tackle Case 0. From (7b), the problem in (12) becomes

\[
\arg\max_{N \in \{N_{\min}, \ldots, N_{\max}\}} \frac{Q_1(\sqrt{2N \rho}, \sqrt{2\gamma^*})}{N}. \tag{33}
\]

To solve (33) we proceed as in Case 3: we relax the problem and introduce the function \( f : (0, \infty) \rightarrow \mathbb{R} \), defined as

\[
f(x) = \frac{\rho Q_1(\sqrt{2x}, \sqrt{2\gamma})}{x}. \tag{34}
\]

This is a positive function with \( \lim_{x \to 0^+} f(x) = \infty \), since \( \lim_{x \to 0^+} Q_1(\sqrt{2x}, \sqrt{2\gamma}) = e^{-\gamma} \), and \( \lim_{x \to \infty} f(x) = 0 \);
f′(x) = \frac{\rho}{x^2} \left( xQ_2(\sqrt{2x}, \sqrt{2\gamma}) - (1 + x)Q_1(\sqrt{2x}, \sqrt{2\gamma}) \right).

(35)

Therefore, f′(x) ≥ 0 if and only if

\[ h(x) = x \left( \frac{Q_2(\sqrt{2x}, \sqrt{2\gamma})}{Q_1(\sqrt{2x}, \sqrt{2\gamma})} - 1 \right) ≥ 1. \quad (36) \]

The ratio of generalized Marcum Q-functions of consecutive orders

\[ g_m(x, \gamma) = \frac{Q_{m+1}(\sqrt{2x}, \sqrt{2\gamma})}{Q_m(\sqrt{2x}, \sqrt{2\gamma})}, \quad m ≥ 1 \quad (37) \]

has been studied in [40], and g_m(x, γ) is greater than one, increasing with γ, and decreasing with M [40, Th. 6]. Furthermore, it can be easily seen that

\[ \lim_{x \to 0^+} g_1(x, \gamma) = \gamma + 1 \quad (38a) \]
\[ \lim_{x \to \infty} g_1(x, \gamma) = 1 \quad (38b) \]
\[ \lim_{\gamma \to 0^+} g_1(x, \gamma) = 1 \quad (38c) \]
\[ \lim_{\gamma \to \infty} g_1(x, \gamma) = \infty. \quad (38d) \]

As to the function h, we have

\[ \lim_{x \to 0^+} h(x) = \lim_{x \to \infty} h(x) = 0 \quad (39) \]

which means that h increases and then decreases with x, achieving a maximum for any γ > 0; the maximum is increasing with γ and tends to infinity when γ → ∞. This framework is summarized in Fig. 13 and is the same analyzed in Case 3, so that (13) is proven also for Case 0.

### References


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